UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 4 (optional)
Spring 2021

1. Transforms and Independence

In this problem, we will make use of multivariate moment generating functions, defined for a random vector $X = (X_1, \dots, X_n)$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ as

$$M_X(t) = \mathbb{E}[e^{t \cdot X}] = \mathbb{E}[e^{\sum_{i=1}^n t_i X_i}].$$

You may assume these m.g.f.'s are unique, i.e. if $M_X(t) = M_Y(t)$ for all t then $X \sim Y$.

Consider the random vector $X = (X_1, X_2, \dots, X_n)$. Show that $X_1, X_2, \dots, X_n$ are independent if and only if for all $t \in \mathbb{R}^n$,

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t_i).$$

2. Jointly Gaussian Equivalence

(a) Compute the m.g.f. of $X \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Use the previous part to show that for some random variables X_i with means μ_i and variances σ_i^2 for $1 \le i \le n$, the following two statements are equivalent:

(i) The $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ are independent Gaussians.

(ii) For every $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, the linear combination

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

is distributed as $\mathcal{N}(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$.

3. Transform Practice

Consider a random variable Z with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \qquad \text{for } |s| < 2.$$

Calculate the following quantities:

(a) The numerical value of the parameter a.

(b) $\mathbb{E}[Z]$.

(c) $\text{var}(Z)$.

4. Chernoff Bound Application: Load Balancing

Here, we will give an application for the Chernoff bound which is instrumental for calculating confidence intervals. However, we will need a slightly more general version of the bound that works for any Bernoulli random variables. For any positive integer n, if $X_1, \dots, X_n$ are i.i.d. Bernoulli, with $P(X_i = 1) = p$, and $S_n = \sum_{i=1}^{n} X_i$, then the following bound holds for $0 \le \varepsilon \le 1$:

$$P\big(S_n > (1+\varepsilon)np\big) \le \exp\Big(-\frac{\varepsilon^2 np}{3}\Big). \tag{1}$$

You may take this as a fact (or try to prove it on your own if you want!).

Here is the setting: there are k (k a positive integer) servers and n users. The simplest load balancing scheme is simply to assign each user to a server chosen uniformly at random (think of the users as "balls" and we are tossing them into server "bins"). By using the union bound, show that with probability at least $1 - 1/k^2$, the maximum load of any server is at most $n/k + 3\sqrt{\ln k}\sqrt{n/k}$. (You may assume that n is much larger than k.)

5. **Gambling Game**

Let's play a game. You stake a positive initial amount of money w_0. You toss a fair coin. If it is heads you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails you lose everything. There is one requirement though, you are not allowed to quit and have to keep playing, by staking all your available wealth, over and over again.

Let W_n be a random variable which is equal to your wealth after n plays.

(a) Find $\mathbb{E}[W_n]$ and show that $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$.

(b) Since $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$, this game sounds like a good deal! But wait a moment!! Where does the sequence of random variables $\{W_n\}_{n\ge 0}$ converge almost surely (i.e. with probability 1) to?