UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 9
Spring 2021

1. **Geometric Sum of Exponentials**

   Let $X_1, X_2, \ldots$ be iid exponentials with parameter λ. If $N \sim \text{Geom}(p)$ taking values on $\{1, 2, \ldots\}$, then show that

   $$\sum_{i=1}^{N} X_i$$

   is exponential and determine its parameter. *Hint:* Consider Poisson thinning.

2. **Arrival Times of a Poisson Process**

   Consider a Poisson process $(N_t, t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let T_i be a random variable which is equal to the time of the i-th arrival.

   (a) Find $\mathbb{E}[T_3 \mid N_1 = 2]$.

   (b) Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.

   (c) Find $\mathbb{E}[T_2 \mid T_3 = s]$.

3. **Illegal U-Turns**

   Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let N be the number of police cars you see before you make a U-turn.

   (a) Find $\mathbb{E}[N]$.

   (b) Let n be a positive integer ≥ 2. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

   (c) Find the expected time that you wait until you make a U-turn.

4. **Bus Arrivals at Cory Hall**

   Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

   (a) Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. (Here, x is a given number, not a random quantity.)

   (b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

   (c) Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

5. **Basketball II**

Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team.

(a) Setup the problem as a CTMC by drawing the chain.

(b) Suppose $\lambda_A = \lambda_B$, and Team A has a head start of $m < k$ points. Find the probability that Team A wins.

(c) Keeping the assumptions from part (b), find the expected time $\mathbb{E}[T]$ it will take for the game to end.