1. Random Graph Estimation

Consider a random graph on $n$ vertices in which each edge appears independently with probability $p$. Let $D$ be the average degree of a vertex in the graph. Compute the maximum likelihood estimator of $p$ given $D$. You may approximate $\text{Binomial}(n, p) \approx \text{Poisson}(np)$.

2. Flipping Coins and Hypothesizing

You flip a coin until you see heads. Let

$$X = \begin{cases} 1 & \text{if the bias of the coin is } q > p. \\ 0 & \text{if the bias of the coin is } p. \end{cases}$$

Let $Y$ be the number of flips until we see heads. Find a decision rule $\hat{X}(Y)$ that maximizes $P[\hat{X} = 1 \mid X = 1]$ subject to $P[\hat{X} = 1 \mid X = 0] \leq \beta$ for $\beta \in [0, 1]$. Remember to calculate the randomization constant $\gamma$.

3. Hypothesis Testing for Uniform Distribution

Assume that

- If $X = 0$, then $Y \sim \text{Uniform}[-1, 1]$.
- If $X = 1$, then $Y \sim \text{Uniform}[0, 2]$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized decision rule $r : [-1, 2] \to \{0, 1\}$ with respect to the criterion

$$\min_{\text{randomized } r : [-1, 2] \to \{0, 1\}} P(r(Y) = 0 \mid X = 1)$$

subject to $P(r(Y) = 1 \mid X = 0) \leq \beta$.

(Note: We will be following the notation used in Walrand and lecture where the probability of false alarm (PFA) is bounded by $\beta$ as opposed to $\alpha$ used in the course notes.)