EECS 126 Probability and Random Processes | UC Berkeley, Fall 2021
Shyam Parekh | December 16, 2021

Final

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 170 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $170 \cdot X\%$ time on the exam and 10 minutes to submit).
- This exam is not open book. You are permitted to use three double-sided handwritten cheat sheets. No calculator or phones allowed.
- Collaboration is prohibited. If caught cheating, you may fail and face disciplinary actions.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		22
Problem 2		18
Problem 3		15
Problem 4		9
Problem 5		11
Problem 6		14
Problem 7		11
Problem 8		15
Total		116

1 Another Potpourri of Probability [4 + 7 + 4 + 7 points]

(a) **Coin Flips [4 points]**

Let X_1, X_2 be i.i.d. Bernoulli(1/2) random variables (i.e. fair coin flips).

Show for this choice of X_1 and X_2 that $H(X_1) + H(X_2) \geq H(X_1 + X_2)$.

(b) **Gaussians [2 + 2 + 3 points]**

Let $X = 2Z_1 + 3Z_2$ and $Y = Z_1 + 2Z_2$, where $Z_1, Z_2 \sim_{\text{iid}} N(0, 1)$.

(i) What is the covariance matrix between X and Y, where the entries are

$$\begin{bmatrix} \text{var}(X) & \text{cov}(X, Y) \\ \text{cov}(Y, X) & \text{var}(Y) \end{bmatrix}?$$

(ii) Find Ł$[X|Y]$.

(iii) Find MMSE$[X|Y]$.

(c) **Poisson MGF [4 points]**

The MGF of $X \sim \text{Poisson}(\lambda)$ is given by $\mathrm{E}[e^{tX}] = e^{\lambda(e^t-1)}$. Using this fact, find the distribution of $X+Y$ where $X \sim \text{Poisson}(\lambda), Y \sim \text{Poisson}(\mu)$, and X is independent of Y.

Note: Finding the distribution without using the MGF will not receive any credit.

(d) **Tom and Jerry [3 + 4 points]**

Tom, Jerry, and 6 of Jerry's other friends are sitting in a room. Outside of the room is a chunk of cheese. At every hour, exactly one of them will get up and exit the room with uniform probability (i.e. if n of them are left, then any one of them will exit with probability $\frac{1}{n}$). Once someone exits the room, they will not return. If Jerry or any one of his friends exits and sees the cheese outside the room, they will eat it completely and leave nothing behind. However, Tom will ignore the cheese.

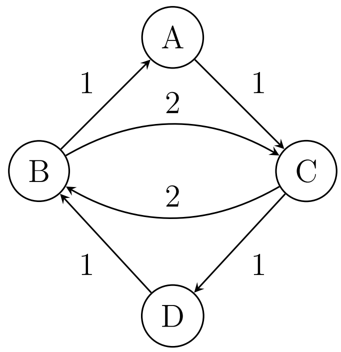
(i) What's the probability that Jerry will get to eat the cheese?

(ii) Now suppose Sohom is also sitting in the room alongside Tom, Jerry, and Jerry's 6 friends. Similar to Tom, Sohom will ignore the cheese once he exits the room. Now what's the probability that Jerry will get to eat the cheese?

2 Markov Chain(s) [8 + 10 points]

(a) **CTMC [4 + 4 points]**

Consider the following CTMC:



(i) Compute its stationary distribution π_{CTMC} using the associated jump chain.

(ii) Compute its stationary distribution π_{CTMC} using uniformization.

Note: Finding π_{CTMC} without using the specified method will not receive any credit.

(b) **DTMC [4 + 6 points]**

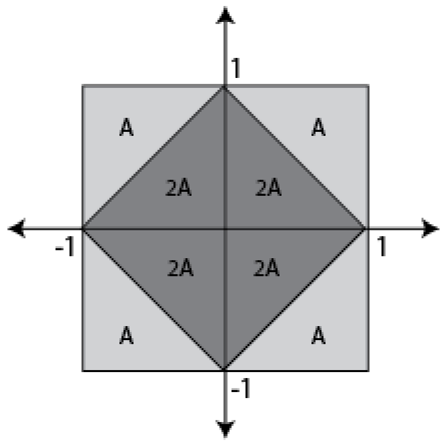
(i) Construct an irreducible discrete time Markov chain with a stationary distribution $(\frac{1}{2}, \frac{1}{2})$. Verify your solution by showing $\pi P = \pi$ where $\pi = (\frac{1}{2}, \frac{1}{2})$.

(ii) Construct an irreducible discrete time Markov chain such that the stationary distribution is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Verify your solution by showing $\pi P = \pi$ where $\pi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

Note: No credit will be given for this question if you do not explicitly show that $\pi P = \pi$.

3 Graphical Density [6 + 3 + 6 points]

Consider the joint density $f_{X,Y}$ shown below:



(a) Find the value of A and determine closed-form expressions for f_X and f_Y.

(b) Compute $\mathrm{E}[X|Y = y]$ for $-1 \leq y \leq 1$.

(c) Are the random variables X and Y independent? Are they uncorrelated?

4 Fishin' Processes [4 + 5 points]

Akshit is fishing and observes that salmon arrive to his fishing hook according to a Poisson process with rate λ_s per minute, and tuna arrive according to a Poisson process with rate λ_t per minute. These two Poisson processes are independent.

(a) What is the probability that at least 2 salmon will arrive in one hour?

(b) Akshit decides to start selling salmon and tuna! He learns that some of the salmon and tuna are poor quality, so with probability 0.1 he will discard a fish that he catches, independent of each other. Suppose Albert is waiting to buy fish from Akshit. Assuming there are no previously caught fish available, how long can Albert expect to wait for the next non-discarded fish?

5 Tennis Distribution [5 + 6 points]

Clark is playing tennis! The rate at which he hits depends on the quality of his tennis balls. Suppose that the tennis ball quality is distributed as $\Lambda \sim \text{Geometric}(p)$, for some fixed $p \in (0, 1)$ and given that $\Lambda = \lambda$, the number of balls he hits is distributed as $X \sim \text{Poisson}(\lambda)$.

Hint: If $f(x) > 0$ for all x, then $\arg\max_x f(x) = \arg\max_x \ln(f(x))$.

(a) What is $\text{MLE}[\Lambda|X]$?

(b) What is $\text{MAP}[\Lambda|X]$?

6 Go Bears! [4 + 4 + 6 points]

Suppose that Cal wins the Big Game with probability p and Stanford with probability $1 - p$, independent of any previous year's result. Your friend at Stanford suggests that $p = \frac{1}{3}$, but you think that $p = \frac{2}{3}$. To decide who's right, you plan to observe the result of three games. Let Y be the number of games Cal wins, and let X be a binary random variable indicating whether you are correct. That is, $X = 0 \iff p = \frac{1}{3}$ and $X = 1 \iff p = \frac{2}{3}$.

Follow the steps to construct a Neyman-Pearson decision rule to maximize $\Pr\{\hat{X} = 1 | X = 1\}$ under the constraint that $\Pr\{\hat{X} = 1 | X = 0\} \leq \frac{1}{3}$, where $\hat{X}$ is the output of the decision rule.

(a) Find the likelihood ratio $L(y)$ for $y \in \{0, 1, 2, 3\}$

(b) Given $X = 0$, find the values that $L(Y)$ takes on and the associated probabilities.

(c) Construct the Neyman-Pearson decision rule.

7 Bacteria-LLSE [11 points]

We have a colony of bacteria with initial population $X \sim \text{Poisson}(\lambda)$. Overnight, each bacterium produces a Poisson (λ) number of offspring independently of the others before it passes away. Let Z be the number of bacteria at beginning of the next day.

Mathematically, we can represent this process by letting $X \sim \text{Poisson}(\lambda)$, $Y_1, Y_2, \ldots \sim_{\text{iid}} \text{Poisson}(\lambda)$ independent of X, and $Z = \sum_{i=1}^{X} Y_i$.

Compute $\mathrm{L}[X|Z]$.

Hint: The law of total variance, $\text{var}(Y) = \mathrm{E}[\text{var}(Y|X)] + \text{var}(\mathrm{E}[Y|X])$, may be helpful.

8 Filter Finale [4 + 5 + 6 points]

Consider the system

$$X_n = aX_{n-1} + V_n$$
$$Y_n = X_n + W_n$$
$$n \geq 1$$

where X_0 is zero mean, $(V_n)_{n\geq 1} \sim_{\text{iid}} N(0, \sigma_v^2)$, and $(W_n)_{n\geq 1} \sim_{\text{iid}} N(0, \sigma_w^2)$, all independent of each other. In class, we have seen the Kalman Filter, which estimates X_n given $(Y_1, \ldots, Y_n)$, or $\hat{X}_{n|n}$, as observations stream in. We now wish to work with the Kalman Predictor, which estimates X_{n+1} given $(Y_1, \ldots, Y_n)$, or $\hat{X}_{n+1|n}$. The update equations are shown below (with one missing part). Note that k_n represents the usual Kalman Filter gain.

$$\hat{X}_{n+1|n} \leftarrow a\hat{X}_{n|n-1} + k_n \cdot 1$$

$$\sigma^2_{n+1|n} \leftarrow a^2\sigma^2_{n|n} + \sigma_v^2$$

$$k_n \leftarrow \sigma^2_{n|n-1} \cdot \left(\sigma^2_{n|n-1} + \sigma_w^2\right)^{-1}$$

$$\sigma^2_{n|n} \leftarrow (1 - k_n)\sigma^2_{n|n-1}$$

(a) Find the linear innovation $\tilde{Y}_n = Y_n - \mathrm{L}[Y_n | Y_1, \ldots, Y_{n-1}]$. Express your answer in terms of Y_n and $\hat{X}_{n|n-1}$.

(b) Show that $a\tilde{Y}_n$ fills blank 1.

(c) Suppose $X_0 = 0, a = 1$, and $\sigma_v^2 = \sigma_w^2 = 2$. Initialize $\hat{X}_{1|0} = 0$ and $\sigma^2_{0|0} = 0$. Use the update equations above to express $\hat{X}_{3|2}$ as a linear function of Y_1 and Y_2.