EECS 126 Probability and Random Processes
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Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 80 minutes to complete the exam. (DSP students with $X\%$ time accomodation should spend $80 \cdot X\%$ time on the exam and 10 minutes to submit).
- This exam is not open book. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		40
Problem 2		9
Problem 3		10
Problem 4		12
Total		72

1 A Potpourri of Probability [7 + 9 + 8 + 8 + 8 points]

(a) **Zhiwei's Anime Collection [2 + 5 points]**

Zhiwei just finished his EECS126 midterm. He decides to watch some anime before going to bed. Zhiwei has 4 animes to choose from: 1) $\mathbb{A}$ttack on Titan, 2) $\mathbb{B}$east King GoLion, 3) $\mathbb{C}$owboy Bebop, and 4) $\mathbb{D}$emon Slayer. Every time he finishes one episode from an anime, he picks the next one to watch randomly from the other 3 animes with equal probability. Assume that Zhiwei never runs out of episodes to watch from any anime.

(i) Find the probability that the **fifth** episode Zhiwei watches is Attack on Titan given that the first episode that he watched was uniformly randomly chosen from the 4 animes.

(ii) Find the probability that the **first** episode he watched was Cowboy Bebop given that the fifth episode he watches is Attack on Titan, and he chooses the first episode uniformly at random from the 4 animes.

(i) The sequences of episodes are symmetric in terms of which anime it is from. Therefore, the probability is just $\frac{1}{4}$.

(ii) Let A:= the first episode he watches is from Cowboy Bebop. Let B:= the fifth episode he watches is from Attack on Titans. Then, $Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)}$. $Pr(A) = \frac{1}{4}$, and $Pr(B) = \frac{1}{4}$ from part (a). Now, let P_n represent that probability that the n-th episode Zhiwei watched is from Attack on Titan given that the first episode he watches is from Cowboy Bebop. Therefore $P_n = \frac{1}{3}(1 - P_{n-1})$. We know that $P_1 = 0$. We can derive that

$$P_2 = \frac{1}{3}$$
$$P_3 = \frac{2}{9}$$
$$P_4 = \frac{7}{27}$$
$$P_5 = \frac{20}{81}$$

Therefore $Pr(A|B) = \frac{20}{81}$.

 Student ID: ____________

(b) **Donut [9 points]**

A point (x, y) is chosen uniformly at random from the region in the xy-plane defined by $\frac{1}{4} \leq x^2 + y^2 \leq 1$. Let X be a random variable representing the x-coordinate of the point. Compute $\mathrm{E}[X]$ and $\mathrm{var}(X)$. (Hint: Think about symmetry and the general equation for a circle.)

By the symmetry of the region, $\mathrm{E}[X] = 0$. Thus, $\mathrm{var}(X) = \mathrm{E}[X^2]$.

Let Y be the random variable representing the y-coordinate of the point, and let R be the random variable representing the distance from the point from the origin. Then note that $X^2 + Y^2 = R^2$, so $\mathrm{E}[X^2] + \mathrm{E}[Y^2] = \mathrm{E}[R^2]$. Moreover, by symmetry, $\mathrm{E}[X^2] = \mathrm{E}[Y^2]$. Thus, we wish to compute $\frac{E[R^2]}{2}$.

We now compute the distribution of R. Notice that $\mathrm{P}(R \leq r) = \frac{\pi\left(r^2 - \frac{1}{4}\right)}{\pi\left(1 - \frac{1}{4}\right)} = \frac{4}{3}r^2 - \frac{1}{3}$, so $f_R(r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{P}(R \leq r) = \frac{8r}{3}$. Thus,

$$\mathrm{E}[R^2] = \int_{\frac{1}{2}}^{1} r^2 f_R(r)\mathrm{d}r = \int_{\frac{1}{2}}^{1} \frac{8r^3}{3}\mathrm{d}r = \left[\frac{2r^4}{3}\right]_{\frac{1}{2}}^{1} = \frac{2}{3} - \frac{1}{24} = \frac{5}{8},$$

and thus $\mathrm{var}(X) = \frac{1}{2} \cdot \frac{5}{8} = \boxed{\frac{5}{16}}$.

(c) **Independence and Correlation [4 + 4 points]**

i) Show that independence implies uncorrelatedness.

ii) Is the converse true? If not, provide a counterexample.

i) $P(X, Y) = P(X)P(Y)$ by definition of independence.

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X = x, Y = y)xydydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(X = x)f(Y = y)dydx \\ &= \left(\int_{-\infty}^{\infty} xf(X = x)dx\right)\left(\int_{-\infty}^{\infty} yf(Y = y)dy\right) \\ &= E[X]E[Y] \end{aligned}$$

$E[XY] = E[X]E[Y] \implies Cov(X, Y) = E[XY] - E[X]E[Y] = 0$

ii) No. Let $X \sim U(-1, 1)$ and $Y = X^2$. Then these random variables are uncorrelated but dependent.

(d) **Exponential Bound [8 points]**

Let $b > 0$ be a constant. Suppose X is a non-negative continuous random variable such that for any $z > 0$, the inequality $\mathrm{P}(X > z) \leq \exp(-\frac{z}{b})$ holds. Show that $\mathrm{E}[X] \leq b$.

$$
\begin{aligned}
\mathrm{E}[X] = \int_0^\infty \mathrm{P}(X > z)dz &= \int_0^\infty \mathrm{P}(X > z)dz \\
&\leq \int_0^\infty \exp(-\frac{z}{b})dz \\
&= \Big[-b\exp(-\frac{z}{b})\Big]_{z=0}^{z=\infty} \\
&= \Big[-b\exp(-\frac{z}{b})\Big]_{z=0}^{z=\infty} \\
&= \Big[0\Big] - \Big[-b\exp(-\frac{0}{b})\Big] \\
&= b
\end{aligned}
$$

(e) **Big Theorem [2 + 2 + 2 + 2 points]**

Consider an infinite sequence of independent coin flips with probability $\frac{1}{4}$ of heads. Let $\{X_n, n \geq 0\}$ be a Markov Chain, with $X_0 = 0$ and $\{X_n, n \geq 1\}$ denoting the total number of heads so far at flip n (inclusive) modulo 3.

i) Find the period of the Markov Chain.

ii) Is this Markov Chain irreducible?

iii) Find the transition probability matrix of this Markov Chain.

iv) Find the invariant distribution of this Markov Chain.

i) Period = 1 because of self-loops in the Markov Chain.

ii) This Markov Chain is irreducible.

iii) $P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \end{bmatrix}$

iv) $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$

2 An Unfair Game [2 + 7 points]

Suppose you're playing a sequence of coin-flipping games, where for the i-th game you flip a fair coin i times. If all i flips turn up heads, you win 2^i dollars. Else, you lose 1 dollar. Let X_i be the amount of money you win or lose in your i-th game (i.e. X_i can be 2^i or -1).

(a) Compute $\mathrm{E}[X_i]$.

(b) Let $S_n = X_1 + \cdots + X_n$. Show that $P(\lim_{n\to\infty} S_n/n = -1) = 1$.

(a) We compute

$$\mathrm{E}[X_i] = \frac{1}{2^i}2^i - \left(1 - \frac{1}{2^i}\right) = \frac{1}{2^i}.$$

(b) Let A_k be the event that in game k you won 2^k dollars. Then

$$\sum_{k=1}^{\infty} \mathrm{P}(A_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

So, by Borel-Cantelli, we know that almost surely only finitely many of the A_k occur. Therefore, with probability one the limit of S_n/n is just $\lim_{n\to\infty} -n/n = -1$.

3 Picking Apples [10 points]

Akshit is picking apples until he gets bored. After every apple picked, he gets bored and leaves with probability p. Each apple has radius $R \sim \mathcal{N}(a, 2)$ and mass $M \sim \mathcal{N}(R^2, 4)$. Let T be the total mass of apples that Akshit picks. Find $E[T]$. (Assume that the probabilities for R and M being negative are negligible.)

Observe that $N \sim Geom(p)$, the number of apples he picks until he's bored. We can use iterated expectation to find $E[T]$.

$$E[T] = E[E[T|N]] = E[E[\sum_{i=1}^{N} M_i]] = E[NE[M_i]]$$

$$E[M_i] = E[E[M_i|R]] = E[R^2] = Var(R) + E[R]^2 = 2 + a^2$$

$$E[T] = (2 + a^2)E[N] = \frac{(2 + a^2)}{p}$$

4 Some Inversions [5 + 7 points]

Given a list containing the integers $\{1, 2, \ldots, n\}$, define an *inversion* to be a pair of integers i and j such that $i < j$ but i appears after j in the list. For example, the list $[3, 1, 4, 2]$ has 3 inversions: $(3, 1)$, $(4, 2)$ and $(3, 2)$. Now, consider a random permutation. Let X denote the number of inversions it has.

(a) Find $\mathrm{E}[X]$.

(b) Inspector Catherine comes along and chooses two positions at random. If the numbers in those positions are not inverted, she is satisfied and does not change the list. However, if the numbers in those positions are out of order, she sorts the entire list. Let X' be the number of inversions after Inspector Catherine is finished. Find $\mathrm{E}[X']$ in terms of $\mathrm{E}[X]$ and $\mathrm{E}[X^2]$.

(a) Let A_{ij} for $1 \le i < j \le n$ be the event that i appears after j and forms an inversion and let X_{ij} be the associated indicator. Note that half of the permutations have i before j and half have it after, so we have

$$\mathrm{E}[X] = \sum_{1 \le i < j \le n} \mathrm{E}[X_{ij}] = \binom{n}{2} \mathrm{P}(A_{ij}) = \frac{1}{2}\binom{n}{2}$$

(b) If our original permutation has X inversions, then with probability $p := X/\binom{n}{2}$ the number of inversions goes to 0 and with probability $1 - p$ the number of inversions remains X. This gives, by Law of Iterated Expectation,

$$\begin{aligned} \mathrm{E}[X'] &= \mathrm{E}[\mathrm{E}[X' \mid X]] \\ &= \mathrm{E}\left[X\left(1 - \frac{X}{\binom{n}{2}}\right)\right] \\ &= \mathrm{E}[X] - \frac{2}{n(n-1)}\mathrm{E}[X^2] \end{aligned}$$