UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Midterm 1 Study Guide
Spring 2022

This is a brief summary of the topics we covered that will be in scope for Midterm 1. However, the scope of the exam will encompass all of the homeworks, discussions, lab, and lecture material; if something is not in this document, it is not necessarily out of scope. Students are expected to understand topics in more depth than they are discussed here.

1 Probability Fundamentals

1. Kolmogorov's axioms: probabilities are nonnegative, probability of at least one possible outcome is 1, for disjoint events A, B, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
2. Conditional probability: $\Pr(A, B) = \Pr(A|B)\Pr(B)$
   (a) Law of total probability: for a partition $B_1, .., , B_n$ of Ω, $\Pr(A) = \sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i)$
   (b) Bayes' rule: $\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$
3. Independence implies uncorrelated; reverse not necessarily true
   (a) X, Y are independent if for all A, B: $\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B)$
   (b) Covariance: $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
   (c) Correlation: $\text{Corr}(X, Y) = \frac{\text{Cov}(X,Y)}{\sigma(X)\sigma(Y)}$ ($-1 \leq \text{Corr}(X, Y) \leq 1$ by Cauchy-Schwarz)

2 Common Tools

1. Inclusion-exclusion: $\Pr(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Pr(A_{i_1} \cap \cdots \cap A_{i_k})$
2. Iterated expectation/tower rule: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$
3. Law of total variance: $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$
4. Convolution: we can solve for the pdf of the sum of independent random variables $Z = X + Y$ as $f_Z(z) = \int_{t=-\infty}^{\infty} f_X(z - t)f_Y(t)dt$ (or analogous summation for discrete)
5. Tail sum: $\mathbb{E}[X] = \sum_{x=0}^{\infty} Pr(X > x)$ for discrete, $\int_{x=0}^{\infty} Pr(X > x)$ for continuous
6. Order statistics: for i.i.d. $X_1, \ldots, X_n$, denote $X^{(i)}$ as the i-th smallest value
   (a) For continuous X_i, $f_{X^{(i)}}(y) = n\binom{n-1}{i-1}F_X(y)^{i-1}(1 - F_X(y))^{n-i}f_X(y)$

3 Problem Solving Techniques

1. Be able to use counting to calculate probabilities (combinations, stars and bars, etc.)
2. Be familiar with indicator variables and using them to calculate expectations/variance
3. Be comfortable with using symmetry to simplify calculations
4. Derived distributions: know how to get distributions for functions of random variables
5. Graphical density: reading a pdf from a graph and performing calculations with it

4 Common Distributions

1. Bernoulli: is 1 w.p. p and 0 otherwise
2. Binomial: sum of i.i.d. Bernoullis
3. Geometric/exponential: exhibit unique memoryless property
   (a) Min of exponentials is exponential with rate $\sum_{j=1}^{n} \lambda_j$; $\mathbb{P}(X_k = \min_i X_i) = \frac{\lambda_k}{\sum_{j=1}^{n} \lambda_j}$
4. Poisson(λ) is the limit of a binomial as $n \to \infty$ and $p \to 0$ and $np \to \lambda$
   (a) Poisson merging: for independent $X \sim \text{Pois}(\lambda), Y \sim \text{Pois}(\mu)$, $X + Y \sim \text{Pois}(\lambda + \mu)$
   (b) Poisson splitting: Poisson(λ) with arrivals dropped independently with probability p is distributed as Poisson(λp); the dropped arrivals form an independent Poisson($\lambda(1-p)$)
5. Gaussian: ubiquitous distribution commonly used for modeling noise
   (a) For independent $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

5 Bounds/Concentration Inequalities

1. Union bound: $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$
2. Markov's inequality: $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$ for nonnegative random variable X and $a > 0$
3. Chebyshev's inequality: $\mathbb{P}(|X - \mathbb{E}[X]| \geq c) \leq \frac{\text{Var}(X)}{c^2}$ (apply Markov's to $|X - \mathbb{E}[X]|$)