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## Midterm 2

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Last Name	First Name	SID
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- You have 10 minutes to read the exam and 100 minutes to complete this exam.
- The maximum you can score is 120, but 100 points is considered perfect.
- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones. No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **A correct answer without justification will receive little, if any, credit.**
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		50
Problem 2		20
Problem 3		20
Problem 4		30
Total		100 (+20)

**Problem 1: Answer these questions briefly but clearly. [30]**

**1. Chernoff Bound [8]**

Suppose  $X$  is Gaussian with mean zero and unit variance. We want to obtain an upper-bound on  $\mathbb{P}(X > t)$  for  $t > 0$ . Use Chernoff's bound which involves  $M_X(s)$ , the MGF of  $X$ , and optimize this bound over  $s$  to make it as tight as possible. Recall that the MGF is defined as  $M_X(s) = \mathbb{E}[e^{sX}]$ .

**2. Coins and Markov Chains [8]**

You lay out  $n$  coins on a table, some of which face heads up, and some of which face tails up, at time zero. Then, at each time-step, with some small probability  $r$ , you do nothing. Otherwise, you pick a coin uniformly at random from the  $n$  coins, and flip it over (so if it was facing heads earlier, it will now face tails). We wish to model  $X_t$ , the number of coins that face heads up at time  $t$ . The process is allowed to continue for  $N$  iterations, for some very large  $N$ , and then we stop the process. Estimate the probability that there are  $k$  coins facing heads.

### 3. Got the Message? [8]

We want to send one of  $2^n$  equally likely messages reliably over a Binary Erasure Channel (BEC) with probability of erasure  $p = 0.5$ . For the sake of simplicity, assume that exactly a fraction  $p$  of the input bits are erased through the channel, though you are not told which ones.

(a) What is a tight upper bound on  $n$  if I am allowed 10,000 independent uses of the channel? Give an argument for why this bound cannot be exceeded.

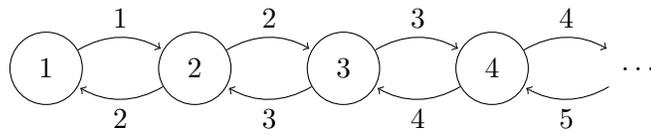
(b) If I use Shannon's random coding scheme, what  $n$  is achievable, assuming that we want to have a probability of success greater than  $1 - 2^{-100}$ ?

### 4. Estimate of Rate for Poisson Process [8]

Consider a Poisson process of unknown rate  $\lambda$ . You have access to observations  $t_1, t_2, \dots, t_n$ , where  $t_i$  represents the time  $i^{\text{th}}$  arrival happens (starting from some arbitrary  $t = 0$ ). Compute the ML estimate of  $\lambda$ .

5. **Infinite CTMC [18]**

Consider a CTMC with states  $\{1, 2, \dots\}$  characterized by the following rate transition diagram below. Is it positive or null recurrent? Provide justification. (Hint:  $\sum_{i=1}^{\infty} \frac{1}{i}$  is a divergent series.)



**Problem 2: Erdős-Rényi Random Graphs & Poisson Processes (The most ambitious crossover in history?) [20]**

Consider a set of  $N$  vertices. For each vertex, assume that there exists a Poisson arrival process of rate  $\lambda$  independent of the arrival processes of the other vertices ( $\lambda$  is the same for all vertices). Once there is an arrival at a vertex, that arrival is routed to one of the other  $N - 1$  vertices, chosen uniformly at random, and *discarded at that vertex*. Assume that this routing is done instantaneously.

Now, let  $T$  be a fixed length of time. We wait for  $T$  seconds, and then draw an edge  $(v, w)$  between vertices  $v$  and  $w$  if and only if the sum of the number of arrivals that are routed from  $v$  to  $w$  and from  $w$  to  $v$  is at least some positive integer  $k$  during the time interval  $[0, T]$ .

- (a) Find the probability that a particular edge exists in the graph. You may have summations in your answer.

- (b) Prove or disprove: The above construction is an Erdős-Rényi random graph, in the sense defined in class. (Remember: For Erdős-Rényi random graphs, each edge exists independently of all others, and the probability of an edge existing is the same across all edges).

### Problem 3: Trick-or-treat Questions for Poisson Processes [20]

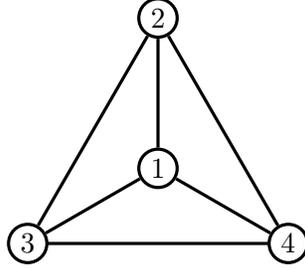
#### 1. Sum of Waiting Times [12]

Suppose that Halloween customers arrive at a spooky costume store in accordance with a Poisson process with rate  $\lambda$ . If the shopkeeper arrives at time  $t$ , compute the expected sum of waiting times of the customers arriving in  $(0, t)$ , conditioned on the fact that there are 10 arrivals in  $(0, t)$ . That is, compute  $\mathbb{E} \left( \sum_{i=1}^{N(t)} (t - S_i) \mid N(t) = 10 \right)$ , where  $S_i$  is the arrival time for the  $i$ th customer and  $N(t)$  is the total number of customers in the interval  $(0, t)$ .

#### 2. Lazy Shopkeeper [8]

A customer arrives at a store at 1 pm. The shopkeeper is lazy and instead of being in the store all the time, visits the store in a Poisson process with rate  $\lambda = 1$  per hour, independent of the arrival time of customer. When the shopkeeper arrives, he gives a discount to the customer linearly proportional to the time since his previous visit to the shop, and the amount of this discount is ten dollars per hour. What is the expected discount the shopkeeper needs to give to the customer? Assume that the shopkeeper has been visiting the store since a very long time ago.

**Problem 4: Joint Random Walk [30]** Alice walks on the following graph randomly, that is, if  $X_k$  is her position at time  $k$ , which is one of the values  $\{1, 2, 3, 4\}$ , then her position at time  $k + 1$  is chosen uniformly at random among the neighbors of  $X_k$  (not including  $X_k$  itself, i.e., you can not stay at the same node on consecutive time steps).



(a) Prove or disprove: The Markov chain  $(X_k : k \geq 0)$  has a unique stationary distribution. [4]

(b) If  $X_0 = 1$ , what is the long time frequency that Alice visits state 2? [6]

(c) Now, assume that Bob joins Alice and they do the above random walk on the graph independently, where  $X_k$  and  $Y_k$  denote Alice's and Bob's position at time  $k$ , respectively. Assume that both of them are initially at state 1, i.e.  $X_0 = Y_0 = 1$ , find  $\lim_{k \rightarrow \infty} \mathbb{P}(X_k = Y_k)$ . [8]

(d) Similar to the previous part, assume that Alice and Bob start at position 1, and walk independently. What is the expectation of the first time that one of them is in the center node 1 and the other one is in one of the other nodes  $\{2, 3, 4\}$ ? [12]  
*[Hint: use symmetry to simplify the problem. Specifically, you can reduce the random walk for each of the two players to 2 states which captures all the information needed for the purpose of finding the expectation.]*