EECS 126 Probability and Random Processes — UC Berkeley, Fall 2021
Shyam Parekh — November 4, 2021

Midterm 2

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 80 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $80 \cdot X\%$ time on the exam and 10 minutes to submit).
- This exam is not open book. No calculator or phones allowed.
- Collaboration is prohibited. If caught cheating, you may fail and face disciplinary actions.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		12
Problem 2		11
Problem 3		9
Problem 4		14
Problem 5		8
Problem 6		13
Total		68

1 A Poisson Chain [2 + 2 + 2 + 3 + 3 points]

Andy runs a discrete-time Markov chain $(X_n)_{n\geq 0}$, starting from $X_0 = 0$. At each time step, if the current state is 0, he samples from a Poisson distribution with a fixed parameter $\lambda \in (0, \infty)$, independent of all other samples, and sets the state to the outcome. Otherwise, if the state is non-zero, he decrements the state by 1. In other words, if $Y \sim \text{Poisson}(\lambda)$, the transition matrix P contains the following values:

$$P_{ij} = \begin{cases} \Pr\{Y = j\} & i = 0 \text{ and } j \geq 0 \\ 1 & i > 0 \text{ and } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

For each question, please provide a brief justification for full credit.

a) What is the state space of this Markov chain?

> Since the Poisson distribution takes values in the natural numbers starting from 0 with positive probability, the state space is $\mathbb{N}_0$.

b) Is this Markov chain irreducible?

> All states communicate with 0, so any two states communicate with each other. Thus, the Markov chain is irreducible.

c) What is the period of this Markov chain?

> Since the Poisson distribution takes the value 0 with positive probability, there is a self-loop in the Markov chain at state 0, which means its period is 1 because the period of a irreducible chain is a class property.

d) Is this Markov chain positive recurrent, null recurrent, or transient? Also, find the expected return time to state 0.

> Starting from state 0, the return time to state 0 is $Y + 1$, where $Y \sim \text{Poisson}(\lambda)$ (one step to transition to Y, and Y steps to transition back to 0). The expected return time is then $\text{E}[Y + 1] = \text{E}[Y] + 1 = \lambda + 1 < \infty$, which means that it's positive recurrent.

e) Does a unique stationary distribution π exist for this Markov chain? If it does, what is π_0?

> By the Big Theorem, since the Markov chain is irreducible and positive recurrent, we know that π exists and is unique, and that π_0 is the inverse of the expected return time to 0. By the previous part, the expected time to return is $\lambda + 1$, so $\pi_0 = \frac{1}{\lambda+1}$.

2 Coin Flipping! [5 + 6 points]

a) One day, Han is bored and decides to flip coins to pass time. These coins are *special*: they will stand up on the thin edge w.p. (with probability) p, land heads w.p. $9p$, and land tails otherwise. Han flips 10 of these coins and observes 5 heads, 3 tails, and 2 coins standing on their edge. Using MLE, what's the most likely value for p?

b) Now, suppose Clark flips a different coin. This coin is fair (shows only heads and tails with equal probability) with probability $\frac{1}{3}$ and otherwise it is a special coin (as defined in the previous problem, now with $p = 0.05$). Given that Clark flips this coin 10 times and observes h heads and $10 - h$ tails, what is the MAP rule for whether or not he is flipping a special coin? Simplify the MAP rule into a single inequality involving h.

a) $L(p) = \binom{10}{5}\binom{5}{3}(9p)^5(1-10p)^3(p)^2$. To find out the MLE estimate for p, we want to maximize $L(p)$. Getting rid of the coefficients, it's same as maximizing $p^7(1-10p)^3$. Taking the derivative with respect to p and setting it to zero, we get $p = 0.07$

b) The special coin has likelihood

$$L_{special}(0.05) = \binom{10}{h}(9 * 0.05)^h(1 - 10 * 0.05)^{10-h}(0.05)^0 * P(\text{choosing a special coin})$$
$$= \binom{10}{h}(0.45)^h(0.5)^{10-h} * \frac{2}{3}$$

while the fair coin has likelihood

$$L_{fair} = \binom{10}{h}(0.5)^h(0.5)^{10-h}(p)^0 * P(\text{choosing a fair coin})$$
$$= \binom{10}{h}(0.5)^h(0.5)^{10-h} * \frac{1}{3}$$

We say that we have the fair coin when

$$\frac{L_{fair}}{L_{special}} > 1.$$

Plugging in the likelihood expressions, we get

$$\frac{L_{fair}}{L_{special}} = \frac{\binom{10}{h}(0.5)^h(0.5)^{10-h} * \frac{1}{3}}{\binom{10}{h}(0.45)^h(0.5)^{10-h} * \frac{2}{3}} = \frac{10^h}{9^h * 2} > 1.$$

Solving for h, we get $h > \log_{10/9} 2$. Since we can only observe a whole number of flips, we conclude the coin is far when $h \geq 7$ and conclude the coin is special when $h \leq 6$.

3 Convergence [9 points]

The Central Limit Theorem says that for $(X_i)_{i=1}^{\infty}$ that are i.i.d. mean zero and variance 1,

$$Z_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{\text{d}} \mathcal{N}(0,1).$$

Show that this limit cannot be upgraded to convergence almost surely. *Hint:* It may help to consider the sequence of random variables $(\sqrt{2}Z_{2n} - Z_n)$.

Suppose $Z_n \xrightarrow{\text{a.s.}} Z \sim \mathcal{N}(0,1)$. Then

$$\sqrt{2}Z_{2n} - Z_n \xrightarrow{\text{a.s.}} (\sqrt{2}-1)Z.$$

However, note that

$$\sqrt{2}Z_{2n} - Z_n = \frac{X_{n+1} + X_{n+2} + \cdots + X_{2n}}{\sqrt{n}},$$

which is equal in distribution to Z_n, and hence converges in distribution to $\mathcal{N}(0,1)$. But almost sure convergence implies convergence in distribution, and the two limits

$$(\sqrt{2}-1)\mathcal{N}(0,1) \neq \mathcal{N}(0,1)$$

disagree in distribution, which is a contradiction. Thus the CLT cannot be upgraded to almost sure convergence. (In fact, by the same argument, it cannot be upgraded to convergence in probability either.)

 Student ID: ____________________

4 Café 126 [4 + 4 + 5 points]

The EECS 126 staff are opening a café! They need your help in planning their business. Suppose that customers arrive according to a Poisson process with rate λ per hour.

a) Let $N(s,t) := N_s - N_t$ be the number of customers that arrive between times t and s. What are $\mathbb{E}[N(3,1) \mid N(6,2)]$ and $\text{Var}[N(3,1) \mid N(6,2)]$?

b) Each customer independently orders exactly one drink, which is a cappuccino with probability 1/2, an espresso with probability 1/3, and a cold brew with probability 1/6. What is the probability that exactly 1 of each drink (cappuccino, espresso, and cold brew) have been sold at the end of T hours?

c) Unfortunately, Café 126 is located adjacent to their competitor, Sunducks. Customers enter Sunducks according to a Poisson Process at rate μ, independent of arrivals to Café 126. What is the probability that, at the time when Café 126 gets their kth customer, Sunducks will have had exactly n customers?

a) Note that $N(2,1)$ is independent of $N(6,2)$ and is distributed according to Poisson(λ). Conditioned on the number of arrivals $N(6,2) = k$, the number of arrivals $N(2,3)$ is given by a binomial distribution Binomial($k, 1/4$). We thus conclude that $\text{E}[N(3,1) \mid N(6,2)] = \lambda + \frac{1}{4}N(6,2))$ and $\text{Var}[N(3,1) \mid N(6,2)] = \lambda + \frac{3}{16}N(6,2)$ from the mean and variance of the Poisson and binomial distributions.

b) By Poisson splitting, we can define independent Poisson processes for each drink. The probability that a Poisson process with parameter λp has exactly 1 arrival in T hours is given by $\lambda p T e^{-\lambda p T}$. Since the split processes are independent from each other, we can multiply the probabilities to give us our answer as $\frac{1}{36}(\lambda T)^3 e^{-\lambda T}$.

c) By Poisson merging, we see the given scenario is equivalent to each customer independently choosing to go to Sunducks with probability $\mu/(\lambda+\mu)$ and Café 126 otherwise. Of the first $n+k$ customers that arrive to either store, the last one must enter Café 126. Thus the probability is

$$\frac{\binom{n+k-1}{k-1}\lambda^k \mu^n}{(\lambda+\mu)^{n+k}}$$

N.B. This precisely the definition of the *negative binomial* distribution.

5 Confident Gambling [4 + 4 points]

It is suspected that a casino is cheating with a die. It is further believed that the probability for a roll resulting in a 6 is set to p, and the probability for each of the other outcomes (1, 2, 3, 4 or 5) is set to $(1-p)/5$. To investigate this, we first estimate p after N rolls by $(X_1 + X_2 + ... + X_N)/N$, where $X_i = 1$ if roll number i has the outcome of 6, and $X_i = 0$, otherwise. We roll the die 100 times, and observe a 6 on 25 of those rolls.

Hints/Facts: The variance of a Bernoulli random variable is at most 1/4. Assume $\Phi^{-1}(0.95) = 1.65$, $\Phi^{-1}(0.975) = 2$ and $\Phi^{-1}(0.99) = 2.32$, where Φ is the CDF of $\mathcal{N}(0,1)$ distribution.

a) Find the 95% confidence interval for p by indicating the numerical values of the endpoints of the interval.

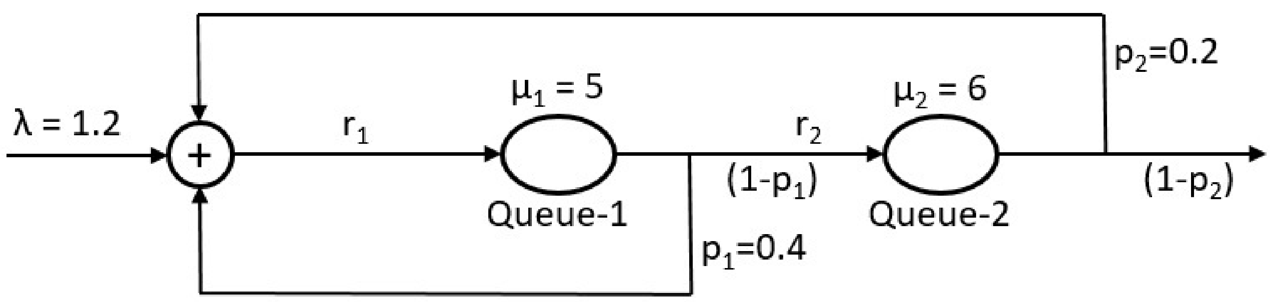
> The two endpoints of the confidence interval are $(X_1 + X_2 + ... + X_N)/N \pm \frac{1}{2\sqrt{N}}\Phi^{-1}(0.975)$. Hence, the 95% confidence interval is $[0.15, 0.35]$.

b) Suppose you want to be sure that the true value of p is within 0.01 of the estimate with at least 0.95 probability. Approximate the minimum value of N required.

> We need $\frac{1}{\sqrt{N}} \leq 0.01$, so $N \geq 10,000$.

6 Jackson [2 + 2 + 3 + 3 + 4 points]

Consider the Jackson network shown in the figure below. Jobs arrive to the network according to a Poisson process with rate $\lambda = 1.2$ jobs/s, and the exponential service rates at Queue-1 and Queue-2 are $\mu_1 = 5$ and $\mu_2 = 6$ jobs/s, respectively. Also, $p_1 = 0.4$ and $p_2 = 0.2$ are the probabilities for a job to rejoin the first queue after service at Queue-1 and Queue-2, respectively. Note that from the time a job enters the network until it leaves the network, it's always at one of the two queues, i.e., routing occurs in zero time.



a) Find the rates r_1 and r_2, the total rates at which jobs enter the two queues, respectively.

b) Find the invariant probability that there are no jobs in the entire network.

c) Find the average number of total jobs in the network under the invariant distribution.

d) What's the average delay for a job (duration from the arrival to and departure from the network) assuming the network is operational for a long time?

e) Is the two-dimensional CTMC associated with this Jackson network reversible? Justify your answer.

a) From the flow conservation equations, we have $r_1 = \lambda + 0.2r_2 + 0.4r_1$ and $r_2 = 0.6r_1$ for Queue-1 and Queue-2, respectively. Solving this system of equations, we get the rates $r_1 = 2.5$ jobs/s and $r_2 = 1.5$ jobs/s.

b) Let (i, j) denote the state that are i jobs at Queue-1 and j jobs at Queue-2. The only state that captures 0 jobs in the network is $(0, 0)$. Due to the Jackson Network Theorem, $\pi((i, j)) = (1 - \rho_1)\rho_1^i(1 - \rho_2)\rho_2^j$, where $\rho_1 = r_1/\mu_1 = 0.5$ and $\rho_2 = r_2/\mu_2 = 0.25$. This gives $\pi((0, 0)) = (1 - \rho_1)\rho_1^0(1 - \rho_2)\rho_2^0 = 0.5 * 0.75 = \frac{3}{8}$.

c) Draw a "box" around the entire network, and apply Little's Law to this "box". Little's Law says arrival rate into this "box" (i.e., $\lambda = 1.2$) times average delay across this "box" (say, $E(D)$) equals the "average" occupancy inside this "box" (say, $E(L)$). Due to the Jackson Network theorem, we know that $E(L) = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} = \frac{4}{3}$.

d) (continuation from previous subpart) Hence, using Little's Law, $1.2E(D) = 4/3$, or $E(D) = 10/9$ s.

e) No. In the two dimension CTMC, it's possible to go from the state (0,0) to the state (1,0), but it's not possible to go from the state (1,0) to the state (0,0). Hence, the Detailed Balance Equations are not satisfied by the two dimensional CTMC.