Midterm 2

Last Name	First Name	SID

- You have 10 minutes to read the exam and 105 minutes to complete this exam.
- The maximum you can score is 125, but 100 points is considered perfect.
- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		48
Problem 2		25
Problem 3		25
Problem 4		27
Total		100 (+25)

Problem 1: Answer these questions briefly but clearly. [48]

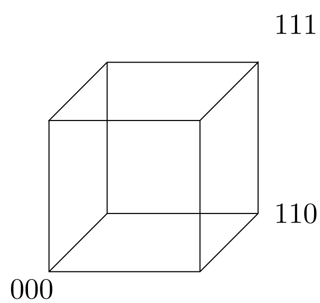
(a) [6] Let $X_1, X_2, \dots$ be a sequence of i.i.d. Uniform($[0,1]$) random variables and define $Y_n = \max\{X_1, \dots, X_n\}$. Prove that $Y_n \xrightarrow{\mathbb{P}} 1$.

(b) [6] Consider a Poisson process $(N(t),\ t \geq 0)$ with rate λ. Does $n^{-1}N(n)$ converge almost surely as $n \to \infty$, and if so, to what? Explain carefully.

(c) [6] Let $S_N = X_1 + \cdots + X_N$, where $N \sim$ Geometric(p) and $X_1, X_2, \dots$ are i.i.d. Exponential(λ) random variables. Using Poisson splitting find the distribution of S_N.

(d) **[6]** Construct an irreducible, transient, aperiodic Markov chain with no self-loops.

(e) **[6]** Consider a 3-dimensional hypercube: the vertices are strings in $\{0,1\}^3$ where two vertices are connected by an edge if and only if they differ by exactly one bit. Start a random walk on the hypercube with starting vertex $X_0 = 110$. At each discrete time step, choose an edge leaving the current vertex uniformly at random and take this edge to the next vertex. What is the probability that the random walk hits 000 before 111?



(f) **[6]** There are 6 people visiting a hospital and we know that exactly one of them is sick. The probability p_i that the i-th person is sick is given by

$$(p_1, p_2, p_3, p_4, p_5, p_6) = (0.5, 0.25, 0.1, 0.05, 0.05, 0.05).$$

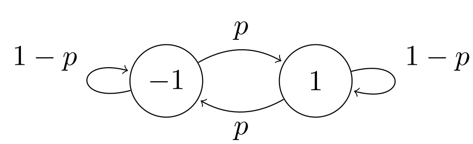
The sickness can be diagnosed by a blood test.

For the first sample, you mix the blood of some of the people and test the mixture, noting that if none of the people in the sample are sick, the test will come up negative. You proceed, mixing and testing, stopping when the sick person has been determined. Give a strategy for mixing the blood of the people, for which the expected number of tests required is at most 2.

(g) **[6]** Consider a Markov chain on the state space $\{-1, 1\}$ such that $P(-1,-1) = P(1,1) = 1-p$ and $P(-1,1) = P(1,-1) = p$ where $p \in (0,1)$. Our goal is to estimate the unknown parameter p. For $i = 1, \ldots, n$, suppose that we observe $Y_i := X_i X_{i+1}$, and we will use the estimator

$$\hat{p} = \frac{\sum_{i=1}^{n}(1 - Y_i)}{2n}.$$

Using the CLT, give a 95% approximate confidence interval for p using the estimator $\hat{p}$.



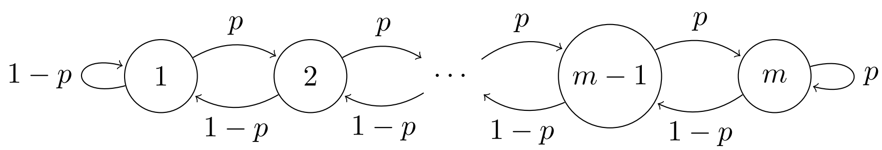
(h) **[6]** Recall that for a sequence $X_1, \ldots, X_n$ of i.i.d. Bernoulli(p) random variables, the typical set, with parameter $\epsilon > 0$, can be written as

$$A_\epsilon^{(n)} = \left\{ (x_1, \ldots, x_n) \in \{0,1\}^n : \left| \sum_{i=1}^{n} x_i - np \right| \leq n\epsilon \right\}.$$

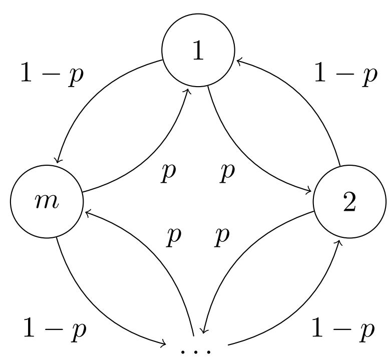
For $n = 10,\ p = 0.6$ and $\epsilon = 0.1$, calculate $\left| A_{0.1}^{(10)} \right|$.

Problem 2: Reversible or Not? [25]

(a) [6] Consider a random walk on $\{1, \ldots, m\}$ given by $P(i, i+1) = p$, $P(i+1, i) = 1-p$ for all $i = 1, \ldots, m-1$, and $P(1,1) = 1-p$, $P(m,m) = p$. Assume that $p \notin \{0,1\}$. Which of the following option(s) is/are true about the chain? (i) It is irreducible. (ii) It is aperiodic. (iii) It is reversible.



(b) [6] Consider a modification of the chain in (b) where the random walk is on a circle: now, $P(i, i+1 \bmod m) = p$ and $P(i, i-1 \bmod m) = 1-p$ for all $i = 1, \ldots, m$. Assume $p \notin \{0,1\}$. Which of the following option(s) is/are true about the chain? (i) It is irreducible. (ii) It is aperiodic.



(c) [6] Does the chain have a unique stationary distribution, and if so, what is it? Under what conditions does it converge to the stationary distribution regardless of the initial distribution?

(d) [7] For what values of p is the chain reversible? (Assume $p \notin \{0, 1\}$.)

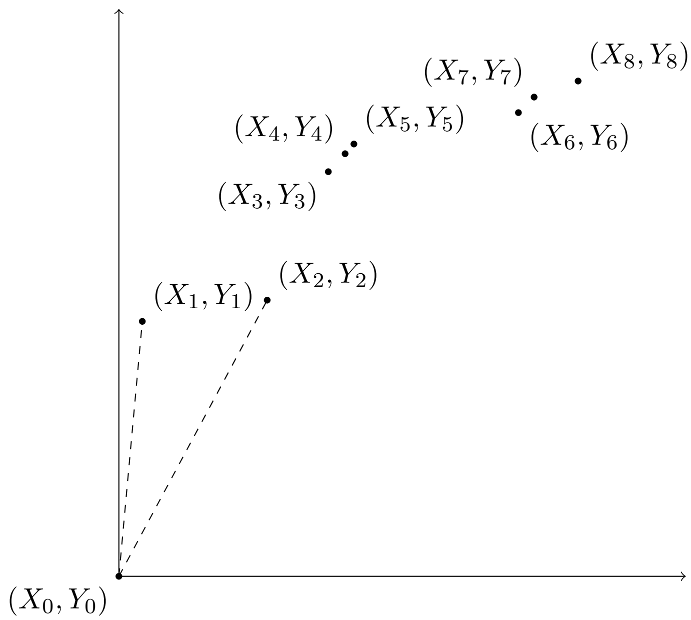
Problem 3: A New Dimension for the Poisson Process [25]

Hint: For all of the following parts, you will find it useful to think about the question in terms of the Poisson processes (recall the PPP: Poisson process perspective).

(a) [12] Let $(U_n)_{n\in\mathbb{N}}$ be i.i.d. Exponential(μ) and $(V_n)_{n\in\mathbb{N}}$ be i.i.d. Exponential(λ) independent of each other, and for each n let $X_n := \sum_{i=1}^{n} U_i$, $Y_n := \sum_{i=1}^{n} V_i$. Think of

$$(X_0, Y_0), (X_1, Y_1), (X_2, Y_2), \ldots$$

as a sequence of points in the plane $\mathbb{R}^2$. What is the probability that the number of points which land in the box $[0, 2]^2$ is at least 2? (You may leave your answer in terms of summations, no need to simplify. Do not leave your answer as an integral.)



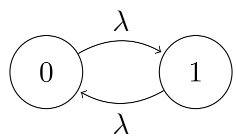
(b) [**4**] What is the probability that the line connecting (X_0, Y_0) to (X_1, Y_1) has slope > 1?

(c) [**9**] What is the probability that the line connecting (X_0, Y_0) to (X_2, Y_2) has slope > 1?

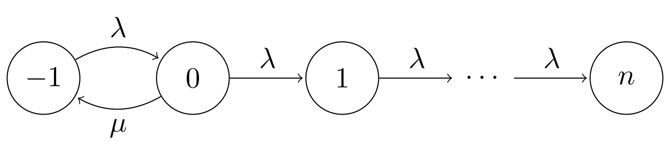
Hint: Think about the merged process.

Problem 4: Time Passing Continuously, Markov! [27]

(a) [**15**] Consider a CTMC on state space $\{0, 1\}$ with $Q(0, 1) = Q(1, 0) = \lambda$. Given that at time 3 there have been a total of 6 times that the CTMC has switched states, what is the probability that the CTMC switched exactly 2 times by time 1?



(b) [**10**] Consider a CTMC on $\{-1, 0, 1, \ldots, n\}$ with $Q(0, -1) = \mu$, and for $i = -1, 0, 1, \ldots, n-1$, $Q(i, i + 1) = \lambda$. What is the expected time, starting from state 0, to hit state n?



(c) [**2**] Was this exam too easy, too hard, or just right? There is no wrong answer.