EECS 126 Probability and Random Processes | UC Berkeley, Spring 2020
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Midterm 2

Last Name	First Name	SID

Rules.

- **Unless otherwise stated, all your answers need to be justified.**
- You may reference your notes, the textbook, and any material that can be found through the course website.
- You may use Google to search up general knowledge. However, **searching up a question is not allowed**.
- **Collaboration with others is strictly prohibited**.
- You have exactly **100** minutes total to both write your exam **AND** submit the exam to Gradescope.
- For any clarifications you have, please create a private Piazza post. We will have a Google Doc that shows our official clarifications.

Grading.

Problem	points earned	out of
Pledge		4
Problem 1		52
Problem 2		18
Problem 3		26
Total		100

1 Assorted Problems

(a) **Is this True?**

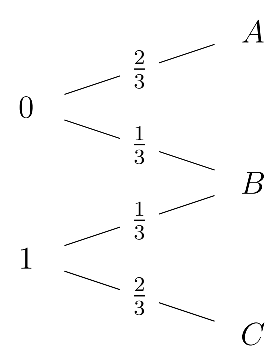
Consider some sequence of random variables $\{X_n\}$. Prove or provide a counterexample for the following.

(i) For this subpart ONLY, take a sequence of random variables $\{X_i\}$ with equal expectation, i.e. $\mathbb{E}[X_i] = a, \forall i$ for some constant a. If variance of X_n converges to 0, then X_n converges in probability to its expectation.

(ii) For any sequence of $\{X_i\}$, convergence in probability to a constant, i.e. $P(|X_n - a| > \epsilon) \to 0$ as $n \to \infty$, implies the variance of X_n goes to 0 as $n \to \infty$.

(b) **Channel Capacity**

Use the formula $C = \max_{p_X} H(Y) - H(Y \mid X)$ to find the capacity of the following channel. You may write your answer in terms of $H_b(p) = -p \log_2 p - (1-p)\log_2(1-p)$.



(c) **Symbols**

We would like to encode the symbols A, B, C, D with some bit sequences. We know that their probabilities are 0.1, 0.2, 0.3, and 0.4, respectively.

(i) What is the entropy of the distribution of symbols? You may leave your answer as an expression.

(ii) Under the optimal encoding procedure, what is the average length of an encoded symbol?

(d) **Chair Game**

Suppose there are three students arranged in a circle. Initially, at time $t = 0$, all three students are sitting down. Every second, a student is chosen uniformly at random. If this student was originally sitting down, then they would stand up, and if they were standing up, they now sit down. What is the expected amount of time that passes until the first time in which all three students are standing up? Leave your answer as a simplified fraction.

(e) **Running Track**

Avishek is pacing around his school running track, which can broken up into 400 1-meter segments. Every time step, he decides with equal probability either to go 1-meter forward, or to turn around and go 1-meter backwards. Given he is currently somewhere along the track, how long will it take for him to next return to his initial position?

(f) **Tipsy Bartenders**

Two bartenders continuously pour drinks a rate of 1L/min into their own respective glasses and serve them according to Poisson Processes with rates 2 drinks/min and 3 drinks/min. Say this process started infinitely long in the past. Given you now choose a bartender **randomly with equal probability**, and you take the next drink the chosen bartender serves, what is the expected volume of your drink?

2 Tacos

A popular restaurant is serving tacos. Each time step, **exactly one** of the following two things occur. With probability p, a new customer comes into the restaurant. With probability $1-p$, a customer waiting in line is served (if there are no customers waiting in line, nothing happens).

(a) Suppose customers are currently in the restaurant. The restaurant cooks like making tacos, but they also want to take a break eventually (when there are no customers in the store). In order for this happen with probability 1, what must be true of p? Briefly justify.

(b) However, the cooks also don't want to wait forever for their break. In order for the expected amount of time they have to wait to be finite, what must be true of p? Briefly justify.

(c) Suppose $p = 1/3$. Sometime in the distant future, Will arrives at the restaurant. What is the probability the restaurant is empty?

3 Basketball

Kevin and Nikita are playing basketball. Baskets are made according to a Poisson process with parameter $\lambda = 2$. Each basket independently has probability $\frac{1}{3}$ of being from Kevin and probability $\frac{2}{3}$ of being from Nikita.

(a) At time $t = 3$, what is the expected lead (in baskets made) that Kevin has over Nikita? This can be negative.

(b) Suppose at time $t = 30$, Kevin had made seven baskets. What is the expected number of baskets Nikita made when Kevin just makes his fifth basket?

(c) Suppose at $t = 4$, three baskets have been made. What is the probability that exactly two baskets were made from $t = 0$ to $t = 2$?

(d) Suppose Kevin only goes for 3-pointers and Nikita only goes for 2-point layups. At time $t = 3$, the **total points scored** is currently 8. Given this, what is the probability that Kevin scored 6 points?