EECS 126 Probability and Random Processes
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UC Berkeley, Spring 2021
April 6, 2021

Midterm 2

Last Name	First Name	SID

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 70 minutes to complete the exam and 10 minutes exclusively for submitting your exam to Gradescope. (DSP students with $X\%$ time accomodation should spend $70 \cdot X\%$ time on the exam and 10 minutes to submit).
- Collaboration with others is strictly prohibited.
- You should not discuss the exam with anyone (this includes your roommate, your parents, social media, reddit, etc.) until 24 hours after the exam concludes (April 7th, 2:10pm).
- You may reference your notes, the textbook, and any material that can be found through the course website. You may use Google to search for general knowledge or use calculators. However, searching for a question is not allowed.
- For any clarifications you have, please create a private Piazza post. We will have a Google Doc that shows our official clarifications.

Problem	points earned	out of
Honor Code		5
Problem 1		15
Problem 2		21
Problem 3		15
Problem 4		26
Problem 5		18
Total		100

Honor Code [5 points]

Please copy the following word for word, and sign afterwards.

By my honor, I confirm that

1. *this work is my own original work;*
2. *I have not and will not discuss this exam with anyone during the exam and for 24 hours after the exam;*
3. *I have not and will not Google/search for any of these exam problems.*

1 Ya∗tze∗ [5 + 10 points]

Consider a 6-sided die, with faces numbered 1-6. The die is weighted so that you are twice as likely to roll an even number as an odd number. Among even numbers, all outcomes are equally likely. Similarly, among odd numbers, all outcomes are equally likely.

(a) Suppose you roll the die $k \gg 1$ times. Approximately how many bits per outcome, on average, are needed to describe the sequence of k independent rolls?

(b) Let $(n_k)_{k\geq 1}$ be a sequence of integers satisfying $\lim_{k\to\infty} k/n_k = B$, where the *bandwidth* $B \geq 0$ is a fixed constant. You would like to communicate the sequence of k rolls from part (a) over n_k uses of a Binary Erasure Channel with erasure probability $p \in [0,1]$ (i.e., a BEC(p)). If you want to make the probability of miscommunication arbitrarily small in the limit as $k \to \infty$, what is the required relationship between p and B?

[Let A denote your answer to part (a), and write your answer in terms of A.]

2 Golden Bear Bets [5 + 7 + 9 points]

Inspired by your GameStop gains, you have bought a share of OSKI stock. This time, you decide to model stock prices with a Markov chain. You believe that every day, prices will increase by \$1 with probability p, decrease by \$1 with probability q (provided the price is \$1 or more), or stay the same. Assume prices take non-negative integer values and $p + q \leq 1$.

(a) Draw out the state transition diagram. Is this chain irreducible?

(b) Under what conditions on p and q is the chain positive recurrent? **Briefly** justify your answer.

(c) Assume $p = q > 0$ and $k \geq 10$. What is the probability that the stock price will reach $k + 30$ before it falls to $k - 10$, if we are starting at state k?

3 Maximizing Uptime [5 + 10 points]

Aditya has two wheels for his unicycle (the second wheel is a backup wheel in case the one in use goes flat, so only one wheel is ever used at a time). When a wheel is being used, the time until it goes flat is exponentially distributed with rate μ. When a wheel goes flat, the other one, if it is not flat, immediately starts being used; additionally, the flat one is taken immediately to the repair shop, where the repair time is exponentially distributed with rate λ (ignore the time it might take to bring it to the repair shop). Assume that only one wheel can be serviced/repaired at a time.

(a) Model the number of operational wheels at time $t \geq 0$ as a CTMC. In particular, draw the transition diagram with states/arrows labeled appropriately, and explicitly state the Q-matrix.

(b) Find the long run probability that Aditya is unable to use his unicycle – i.e., both wheels are flat.

4 Poisson Jobs [6 + 6 + 6 + 8 points]

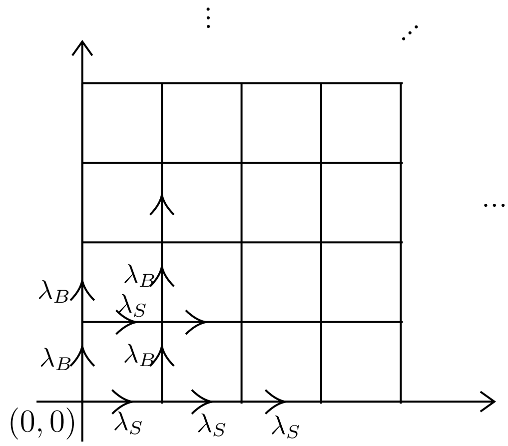

Figure 1: A CTMC where (X_t, Y_t) is the state at time t; vertices represent states.

Amazon receives deep learning jobs from Berkeley and Stanfurd according to independent Poisson processes with respective rates λ_B and λ_S. For $t \geq 0$, Let X_t denote the total number of jobs received from Stanfurd up to and including time t, and let Y_t denote the total number of jobs received from Berkeley up to and including time t. Although not needed for this problem, you can picture the pair (X_t, Y_t) as evolving according to a CTMC, as in Figure 1.

(a) At any given instant, what is the probability that the next three jobs received by Amazon are all from Berkeley?

(b) Suppose you inspect the system at some time $t \gg 0$, assumed to be infinitely far in the future. What is the expected time between the previous job and the next job?

(c) Suppose Amazon has received 2 jobs from Berkeley and 3 jobs from Stanfurd by time t. What is the expected time at which Amazon receives the 10th job from Berkeley?

(d) Again, suppose Amazon has received 2 jobs from Berkeley and 3 jobs from Stanfurd at time t. What is the expected number of jobs Amazon received from Stanfurd when it has received 10 jobs from Berkeley?

5 This Question is Almost Surely Solvable. Probably. [6 + 4 + 8 points]

(a) Let $(S_n)_{n\geq 1}$ be i.i.d. non-negative random variables, with finite mean $\mathbb{E}[S_n] = \mu < +\infty$. Consider an arrival process $(N_t)_{t\geq 0}$ (a continuous-time counting process, but not necessarily a Poisson process, which would require the S_n's to be exponentials) which has inter-arrival times equal to the S_n's. That is, the first arrival comes at time $T_1 = S_1$, the second arrival comes at time $T_2 = T_1 + S_2 = S_1 + S_2$, and so forth, so that the kth arrival comes at time $T_k = \sum_{n=1}^{k} S_n$.

Argue that $\lim_{t\to\infty} N_t = +\infty$ almost surely.

(b) For the setting of part (a), compute $\lim_{t\to\infty} \frac{N_t}{N_t+1}$ and describe its mode of convergence.

(c) For the setting of part (a), compute $\lim_{t\to\infty} N_t/t$, and describe its mode of convergence. [*Hint:* For each $t \geq 0$, we have $T_{N_t} \leq t \leq T_{N_t+1}$. Use this to "sandwich" N_t/t and apply an appropriate law of large numbers.]