UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 12
Spring 2022

1. **Poisson Process MAP**

Customers arrive to a store according to a Poisson process of rate 1. The store manager learns of a rumor that one of the employees is sending every other customer to the rival store. Refer to hypothesis $X = 1$ as the rumor being true, that one of the employees is sending every other customer arrival to the rival store and hypothesis $X = 0$ as the rumor being false, where each hypothesis is equally likely. Assume that at time 0, there is a successful sale. After that, the manager observes $S_1, S_2, \ldots, S_n$ where n is a positive integer and S_i is the time of the ith subsequent sale for $i = 1, \ldots, n$. Derive the MAP rule to determine whether the rumor was true or not.

2. BSC: MLE & MAP

You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0, 0.5]$.

(a) Assume you observe the input and the output of the link. How do you find the MLE of ϵ?

(b) You are told that the inputs are i.i.d. bits that are equal to 1 with probability 0.6 and to 0 with probability 0.4. You observe n outputs (n is a positive integer). How do you calculate the MLE of ϵ?

(c) The situation is as in the previous case, but you are told that ϵ has PDF $4 - 8x$ on $[0, 0.5)$. How do you calculate the MAP of ϵ given n outputs? You may leave your answer in terms of quadratic equation to be solved.

3. Bayesian Estimation of Exponential Distribution

We have already learned about MLE (non-Bayesian perspective) and MAP (Bayesian perspective). In this problem, we will introduce the fully Bayesian approach to statistical estimation.

Suppose that X is an exponential random variable with unknown rate Λ (Λ is a random variable). As a Bayesian practitioner, you have a prior belief that Λ is equally likely to be λ_1 or λ_2.

You collect one sample X_1 from X.

(a) Find the posterior distribution $\mathbb{P}(\Lambda = \lambda_1 \mid X_1 = x_1)$.

(b) If we were using the MLE or MAP rule, then we would choose a single value λ for Λ; this is sometimes called a *point estimate*. This amounts to saying X has the exponential distribution with rate λ.

In the Bayesian approach, we will not use a point estimate. Instead, we will keep the full information of the posterior distribution of Λ, and we compute the distribution of X as

$$f_X(x) = \sum_{\lambda \in \{\lambda_1, \lambda_2\}} f_{X|\Lambda}(x \mid \lambda)\mathbb{P}(\Lambda = \lambda \mid X_1 = x_1).$$

Notice that in the Bayesian approach, we do not necessarily have an exponential distribution for X anymore. Compute $f_X(x)$ in closed-form.

(c) You might guess from the previous part that the fully Bayesian approach is often computationally intractable. This is one of the main reasons why point estimates are common in practice.

Compute the MAP estimate for Λ and calculate $f_X(x)$ again using the MAP rule.

4. **Hypothesis Testing for Bernoulli Random Variables**

Assume that

- If $X = 0$, then $Y \sim \text{Bernoulli}(1/4)$.
- If $X = 1$, then $Y \sim \text{Bernoulli}(3/4)$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized *decision rule* $r : \{0,1\} \to \{0,1\}$ with respect to the criterion

$$\begin{aligned} \min_{\text{randomized } r:\{0,1\}\to\{0,1\}} \quad & \mathbb{P}\big(r(Y) = 0 \;\big|\; X = 1\big) \\ \text{s.t. } & \mathbb{P}\big(r(Y) = 1 \;\big|\; X = 0\big) \leq \beta, \end{aligned}$$

where $\beta \in [0,1]$ is a given upper bound on the false positive probability.

(Note: We will be following the notation used in Walrand and lecture where the probability of false alarm (PFA) is bounded by β as opposed to α used in the course notes.)

5. **BSC Hypothesis Testing**

Consider a BSC with some error probability $\epsilon \in [0.1, 0.5)$. Given n inputs and outputs (x_i, y_i) of the BSC, solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that n is very large and use the CLT.

Hint: The null hypothesis is $\epsilon = 0.1$. The alternate hypothesis is $\epsilon > 0.1$, which is a **composite hypothesis** (this means that under the alternate hypothesis, the probability distribution of the observation is not completely determined; compare this to a **simple hypothesis** such as $\epsilon = 0.3$, which *does* completely determine the probability distribution of the observation). The Neyman-Pearson Lemma we learned in class applies for the case of a simple null hypothesis and a simple alternate hypothesis, so it does not directly apply here.

To fix this, fix some specific $\epsilon' > 0.1$ and use the Neyman-Pearson Lemma to find the optimal hypothesis test for the hypotheses $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$. Then, argue that the optimal decision rule does not depend on the specific choice of ϵ'; thus, the decision rule you derive will be *simultaneously* optimal for testing $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$ for all $\epsilon' > 0.1$.

6. Gaussian Hypothesis Testing

Consider a hypothesis testing problem that if $X = 0$, you observe a sample of $\mathcal{N}(\mu_0, \sigma^2)$, and if $X = 1$, you observe a sample of $\mathcal{N}(\mu_1, \sigma^2)$, where $\mu_0, \mu_1 \in \mathbb{R}$, $\sigma^2 > 0$. Find the Neyman-Pearson test for false alarm $\beta \in (0, 1)$, that is, $\mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \beta$.