1. **Gaussian Estimation**

Let $Y = X + Z$ and $U = X - Z$, where $X$ and $Z$ are i.i.d. $\mathcal{N}(0, 1)$.

(a) Find the joint distribution of $U$ and $Y$.

(b) Find the MMSE of $X$ given the observation $Y$, call this $\hat{X}(Y)$.

(c) Let the estimation error $E = X - \hat{X}(Y)$. Find the conditional distribution of $E$ given $Y$. 
2. Noisy Guessing

Let $X$, $Y$, and $Z$ be i.i.d. with the standard Gaussian distribution. Find $E[X \mid X + Y, X + Z, Y - Z]$.

*Hint:* Argue that the observation $Y - Z$ is redundant.
3. Gaussian Sine

Let $X, Y, Z$ be jointly Gaussian random variables with covariance matrix

$$
\begin{bmatrix}
4 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 4 \\
\end{bmatrix}
$$

and mean vector $[0, 2, 0]$. Compute $E[(\sin X)Y(\sin Z)]$. 

*Hint:* Condition on $(X, Z)$. 

4. Gaussian Random Vector MMSE

Let

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix}
\sim \mathcal{N}
\left(
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\right)
\]

be a Gaussian random vector.

Let

\[W = \begin{cases}
1, & \text{if } Y > 0 \\
0, & \text{if } Y = 0 \\
-1, & \text{if } Y < 0
\end{cases}\]

be the sign of \(Y\). Find \(E[WX \mid Y]\). Is the LLSE the same as the MMSE?
5. Stochastic Linear System MMSE

Let \((V_n, n \in \mathbb{N})\) be i.i.d. \(\mathcal{N}(0, \sigma^2)\) and independent of \(X_0 = \mathcal{N}(0, u^2)\). Let \(|a| < 1\). Define

\[X_{n+1} = aX_n + V_n, \quad n \in \mathbb{N}.
\]

(a) What is the distribution of \(X_n\), where \(n\) is a positive integer?

(b) Find \(\mathbb{E}[X_{n+m} \mid X_n]\) for \(m, n \in \mathbb{N}, m \geq 1\).

(c) Find \(u\) so that the distribution of \(X_n\) is the same for all \(n \in \mathbb{N}\).
6. Error of the Kalman Filter for a Linear Stochastic System

The linear stochastic system

\[
\begin{bmatrix}
X_{1,k+1} \\
X_{2,k+1}
\end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X_{1,k} \\
X_{2,k}\end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_k, \quad k \geq 0,
\]

starts from an arbitrary (known) initial condition \(\begin{bmatrix} x_{1,0} \\
x_{2,0}\end{bmatrix}\) and the system noise variables \((w_k, k \geq 0)\) are i.i.d. normal with mean 0 and variance 1.

The state variables are not directly observable. However, we can observe

\[Y_k = X_{1,k} + X_{2,k}, \quad k \geq 0.\]

Let \(\hat{X}_{k|k}\) denote the minimum mean square error estimator of \(X_k = \begin{bmatrix} X_{1,k} \\
X_{2,k}\end{bmatrix}\) given \((Y_0, \ldots, Y_k)\).

Determine the asymptotic behavior of the covariance matrix of the estimation error.

\textit{Note}: This problem needs thought. Note that there is no observation noise, so the assumption used in the derivation of the Kalman filter equations, that the covariance matrix of the observation noise is positive definite, is no longer valid.
7. Random Walk with Unknown Drift [Optional]

Consider a random walk with unknown drift. The dynamics are given, for \( n \in \mathbb{N} \), as

\[
\begin{align*}
X_1(n+1) &= X_1(n) + X_2(n) + V(n), \\
X_2(n+1) &= X_2(n), \\
Y(n) &= X_1(n) + W(n).
\end{align*}
\]

Here, \( X_1 \) represents the position of the particle and \( X_2 \) represents the velocity of the particle (which is unknown but constant throughout time). \( Y \) is the observation. \( V \) and \( W \) are independent Gaussian noise variables with mean zero and variance \( \sigma^2_V \) and \( \sigma^2_W \) respectively.

(a) Write down the dynamics of the system in matrix-vector form and write down the Kalman filter recursive equations for this system.

(b) Let \( k \) be a positive integer. Compute the prediction \( \mathbb{E}(X(n+k) \mid Y^{(n)}) \), where \( Y^{(n)} \) is the history of the observations \( Y_0, \ldots, Y_n \), in terms of the estimate \( \hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)}) \).

(c) Now let \( k = 1 \) and compute the smoothing estimate \( \mathbb{E}(X(n) \mid Y^{(n+1)}) \) in terms of the quantities that appear in the Kalman filter equation.

\textit{Hint:} Use the law of total expectation

\[
\begin{align*}
\mathbb{E}(X(n) \mid Y^{(n+1)}) &= \mathbb{E}[\mathbb{E}(X(n) \mid X(n+1), Y^{(n+1)}) \mid Y^{(n+1)}],
\end{align*}
\]

as well as the \textit{innovation}

\[
\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].
\]