UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 2
Spring 2022

1. **Among Us**

In the game of Among Us, there are 9 players. 4 of them are imposters and 5 of them are crewmates. There is also a deck of 17 cards containing 11 "sabotage" cards and 6 "task" cards. Imposters want to play sabotage cards, and crewmates want to play task cards. Here's how the play proceeds.

- A captain and a first mate are chosen uniformly at random from the 9 players.
- The captain draws 3 cards from the deck and gives 2 to the first mate, discarding the third.
- The first mate chooses one to play.

Now suppose you are the first mate, but the captain gave you 2 sabotage cards. Being a crewmate, you wonder, did the captain just happen to have 3 sabotage cards, or was the captain an imposter who secretly discarded a task card. In this scenario, what's the probability that the captain is an imposter? Let's assume that imposter captains always try to discard task cards, and crewmate captains always try to discard sabotage cards.

2. **Variance**

If $X_1, \ldots, X_n$, where $n \in \mathbb{Z}_{>0}$, are i.i.d. random variables with zero-mean and unit variance, compute the variance of $(X_1 + \cdots + X_n)^2$. You may leave your answer in terms of $\mathbb{E}[X_1^4]$, which is assumed to be finite.

3. **Lightbulbs**

Consider an $n \times n$ array of switches. Each row i of switches corresponds to a single lightbulb L_i, so that L_i lights up if at least i switches in row i are flipped on. All of the switches start in the "off" position, and each are flipped "on" with probability p, independently of all others. What is the expected number of lightbulbs that will be lit up? Express your answer in closed form without any summations.

4. **Compact Arrays**

Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

$$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

After making the array compact, it now looks like

$$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is non-zero (assume that the array is indexed starting from 1). Let X be a random variable which is equal to the index that the ith entry has been moved after making the array compact. Calculate $\mathbb{E}[X]$ and $\text{var}(X)$.

5. **Unit Circle on a Grid**

A circle of unit radius is thrown on an infinite sheet of graph paper that has square grids of unit length on each side. Find the expected number of vertex points of the grid that fall in the circle.

6. **Random Walk on a Circle**

Suppose we have n points labeled $\{1, 2, \ldots, n\}$ around a circle. An ant starts at point 1, and at each second has an equal probability of moving clockwise or counterclockwise to an adjacent point. For each point $k \in \{2, 3, \ldots, n\}$, find the probability that the first time the ant lands at k, it has visited all other points already.

Hint: Try simulating this problem. What do you observe?