UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 6
Spring 2022

1. Midterm

Solve all of the problems on the midterm again (including the ones you got correct).

2. Convergence in Probability

Let $(X_n)_{n=1}^{\infty}$, be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences $(Y_n)_{n=1}^{\infty}$ converge in probability to some limit.

(a) $Y_n = \prod_{i=1}^{n} X_i$.

(b) $Y_n = \max\{X_1, X_2, \dots, X_n\}$.

(c) $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

3. More Almost Sure Convergence

(a) Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

(b) Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \;\big|\; Y = y\big) = 1.$$

Does $(X_n)_{n=1}^{\infty}$ converge a.s.?

(c) Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$ and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

(d) Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$?

4. Huffman Questions

Consider a set of n objects. Let $X_i = 1$ or 0 accordingly as the i-th object is good or defective. Let $X_1, X_2, \dots, X_n$ be independent with $\mathbb{P}(X_i = 1) = p_i$; and $p_1 > p_2 > \cdots > p_n > 1/2$. We are asked to determine the set of all defective objects. Any yes-no question you can think of is admissible.

(a) Propose an algorithm based on Huffman coding in order to identify all defective objects.

(b) Suppose the worst case scenario happens and we have to ask the maximum number of questions. What (in words) is the last question we should ask? And what two sets are we distinguishing with this question?

5. **Mutual Information and Channel Coding [Optional]**

The **mutual information** of X and Y is defined as

$$I(X;Y) := H(X) - H(X \mid Y)$$

Here, $H(X \mid Y)$ denotes the **conditional entropy** of X given Y, which is defined as:

$$\begin{aligned} H(X \mid Y) &= \sum_{y \in \mathcal{Y}} p_Y(y) H(X \mid Y = y) \\ &= \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{x \in \mathcal{X}} p_{X|Y}(x \mid y) \log_2 \frac{1}{p_{X|Y}(x \mid y)} \end{aligned}$$

The interpretation of conditional entropy is the average amount of uncertainty remaining in the random variable X after observing Y. The interpretation of mutual information is therefore the amount of information about X gained by observing Y.

The channel coding theorem says that if X is passed into the channel and Y is received, then the capacity of the channel is

$$C = \max_{p_X} I(X;Y) = \max_{p_X} H(X) - H(X \mid Y)$$

(a) Let X be the roll of a fair die and $Y = \mathbf{1}\{X \geq 5\}$. What is $H(X \mid Y)$?

(b) Suppose the channel is a noiseless binary channel, i.e. $X \in \{0, 1\}$ and $Y = X$. Use the theorem to find C.

(c) Consider a binary erasure channel with probability of erasure p. Use the theorem to find C.

Hint: To find the optimal p_X, it is helpful to let $p_X(1) = P(X = 1) = \alpha$.