1. Compression of a Markov Chain

Consider an irreducible Markov chain \((X_n)_{n \in \mathbb{N}}\) as shown below.

\[
\begin{array}{c}
\text{0} \\
\uparrow \\
\text{1}
\end{array} \quad \text{with}
\begin{array}{cc}
p & 1 - p \\
1 - p & p
\end{array}
\]

Suppose \(X_0 \sim B(\frac{1}{2})\) and \(p \neq \frac{1}{2}\). A naive scheme to keep track of the states would be to note down each bit individually. Propose a scheme to represent \((X_0, X_1, \ldots, X_n)\) in fewer than \(n\) bits in expectation as \(n \to \infty\), and state how many bits that would take.
2. Fly on a Graph

A fly wanders around on a graph $G$ with vertices $V = \{1, \ldots, 5\}$, shown in Figure 2.

![Graph diagram](image)

Figure 1: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node $i$ at time $n$, then it chooses one of its neighbors $j$ of $i$ uniformly at random, and then wanders to node $j$ at time $n+1$. For times $n = 0, 1, 2, \ldots$, let $X_n$ be the fly’s position at time $n$. Argue that $\{X_n, n \in \mathbb{N}\}$ is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very lazy, so they don’t move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node $i$ at time $n$, it chooses uniformly from all neighbors of node $i$ except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let $Y_n$ be the fly’s position at time $n$. Following the story in the previous part with the professor and window, once the fly reaches node 2 and 5, it remains at that state. Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains and define new states so that it is a valid Markov chain.
3. Finite Random Walk

(a) Assume $0 < p < 1$. Find the stationary distribution. *Hint:* Let $q = 1 - p$ and $\rho = \frac{p}{q}$, but be careful when $\rho = 1$.

(b) Find the limit of $\pi_0$ and $\pi_{k-1}$, the stationary distribution probability for state 0 and $k - 1$, as $k \to \infty$. 
4. Choosing Two Good Movies

You have a database of a countably infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 1, 2, 3, 4, 5\}$ and you want to find two movies such that the sum of their rating is greater than 7.5. Assume that you choose movies from the database one by one and keep the movie with the highest rating so far. You stop when you find that the sum of the ratings of the current movie you have chosen and the movie with the highest rating among all the previous movies is greater than 7.5.

(a) Define an appropriate Markov chain and use the first step equations in order to find the expected number of movies you will have to choose.

(b) Now assume that the ratings of the movies are uniformly distributed in the interval $[0, 5]$. Write the first step equations for the expected number of movies you will have to choose in this case.
5. The Cut Property and Reversible Markov Chains

(a) For an irreducible Markov chain at stationarity, show that the flow-in equals flow-out relationship holds for any cut of the Markov chain. A cut of a Markov chain is a partition of the states into two disjoint subsets. **Hint**: To solve this problem, try induction on the size of one of the subsets of the cut and write out the flow equations at each step.

(b) The state diagram of Markov Chain is a directed graph, but can be converted to an undirected graph in the following manner.

- For every directed edge in the state diagram, use an undirected edge in the undirected graph.
- If A has a directed edge to B and vice versa, there should still only be one undirected edge.

Assuming the Markov chain is irreducible and positive recurrent, a sufficient condition for the detailed balance equations to hold is that the resulting graph is a tree. Explain, in words, why this is true.
6. Markov Chains with Countably Infinite State Space

(a) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:

$$
\begin{array}{c}
\begin{array}{c}
\cdots \\
1 \\
\end{array} \\
\begin{array}{c}
\frac{1}{3} \\
\frac{2}{2i-1} \\
\frac{1}{2} \\
\end{array} \\
\begin{array}{c}
i \\
\frac{i}{2(i+1)} \\
\frac{i}{2(i+1)+2} \\
\end{array} \\
\begin{array}{c}
i+1 \\
\frac{1}{2(i+1)+2} \\
\frac{1}{2} \\
\end{array} \\
\begin{array}{c}
\cdots \\
\frac{3}{4} \\
\frac{1}{2} \\
\end{array}
\end{array}
$$

Show that this Markov chain has no stationary distribution.

(b) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:

$$
\begin{array}{c}
\begin{array}{c}
\cdots \\
1 \\
\end{array} \\
\begin{array}{c}
\frac{a}{b} \\
1-a \\
\end{array} \\
\begin{array}{c}
i \\
\frac{a}{b} \\
1-a-b \\
\end{array} \\
\begin{array}{c}
i+1 \\
\frac{a}{b} \\
1-a-b \\
\end{array} \\
\begin{array}{c}
\cdots \\
1-a \\
\end{array}
\end{array}
$$

Assume that $0 < a < b$ and $0 < a + b \leq 1$. Show that the probability distribution given by

$$
\pi(i) = \left(\frac{a}{b}\right)^{i-1} \left(1 - \frac{a}{b}\right), \text{ for } i \in \mathbb{Z}_{>0},
$$

is a stationary distribution of this Markov chain.