UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 9
Spring 2022

1. Poisson Practice

Let $(N(t), t \geq 0)$ be a Poisson process with rate λ. Let T_k denote the time of k-th arrival, for $k \in \mathbb{N}$, and given $0 \leq s < t$, we write $N(s,t) = N(t) - N(s)$. Compute:

(a) $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

(b) $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

(c) $\mathbb{E}(T_2 \mid N(2) = 1)$.

2. Illegal U-Turns

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let N be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let n be a positive integer ≥ 2. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.

3. Arrival Times of a Poisson Process

Consider a Poisson process $(N(t), t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let T_i be a random variable which is equal to the time of the i-th arrival.

(a) Find $\mathbb{E}[T_3 \mid N(1) = 2]$.

(b) Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.

(c) Find $\mathbb{E}[T_2 \mid T_3 = s]$.

4. **Basketball II**

Captain America and Superman are playing an untimed basketball game in which the two players score points according to independent Poisson processes. Captain America scores points according to a Poisson process with rate λ_C and Superman scores points according to a Poisson process with rate λ_S. The game is over when one of the players has scored k more points than the other player.

(a) Suppose $\lambda_C = \lambda_S$, and Captain America has a head start of $m < k$ points. Find the probability that Captain America wins.

(b) Keeping the assumptions from part (a), find the expected time $\mathbb{E}[T]$ it will take for the game to end.

5. **System Shocks**

For a positive integer n, let $X_1, \ldots, X_n$ be independent exponentially distributed random variables, each with mean 1. Let $\gamma > 0$.

A system experiences shocks at times $k = 1, \ldots, n$. The size of the shock at time k is X_k.

(a) Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

(b) Suppose instead that the effect of the shocks is cumulative, i.e., the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

6. Metropolis-Hastings

This problem proves properties of the **Metropolis-Hastings Algorithm**, which you will see in lab.

Recall that the goal of MH was to draw samples from a distribution $p(x)$. The algorithm assumes we can compute $p(x)$ up to a normalizing constant via $f(x)$, and that we have a proposal distribution $g(x, \cdot)$. The steps are:

- Propose the next state y according to the distribution $g(x, \cdot)$.
- Accept the proposal with probability

$$A(x, y) = \min\left\{1, \frac{f(y)}{f(x)} \frac{g(y, x)}{g(x, y)}\right\}.$$

- If the proposal is accepted, then move the chain to y; otherwise, stay at x.

(a) The key to showing why Metropolis-Hastings works is to look at the **detailed balance equations**. Suppose we have a finite irreducible Markov chain on a state space $\mathcal{X}$ with transition matrix P. Show that if there exists a distribution π on $\mathcal{X}$ such that for all $x, y \in \mathcal{X}$,

$$\pi(x)P(x, y) = \pi(y)P(y, x),$$

then π is a stationary distribution of the chain (i.e. $\pi P = \pi$).

(b) Now return to the Metropolis-Hastings chain. What is $P(x, y)$ in this case? For simplicity, assume $x \neq y$.

(c) Show $p(x)$, our target distribution, satisfies the detailed balance equations with $P(x, y)$, and therefore is the stationary distribution of the chain.

(d) If the chain is aperiodic, then the chain will converge to the stationary distribution. If the chain is not aperiodic, we can force it to be aperiodic by considering the **lazy chain**: on each transition, the chain decides not to move with probability $1/2$ (independently of the propose-accept step). Explain why the lazy chain is aperiodic, and explain why the stationary distribution is the same as before.