EECS 126 Probability and Random Processes — UC Berkeley, Spring 2024
Jiantao Jiao — April 4, 2024

Midterm 2 Review

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- **All work you want graded should be on the fronts of the sheets in the space provided. Back sides may be used for scratch work, but will not be scanned/graded.**
- You have 10 minutes to read the exam and 70 minutes to complete the exam. (DSP students with $X\,\%$ time accommodation should spend $10 \cdot X\,\%$ time on reading and $70 \cdot X\,\%$ time on completing the exam).
- This exam is closed-book. You may reference one double-sided handwritten sheet of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.

1 Convergence [24 points]

For each of the following subparts, please answer (i) whether X_n converges in probability and (ii) whether X_n converges almost surely. Make sure to respond to the above two questions explicitly. In each case, if there is convergence, find and prove what X_n converges to. If there is no convergence guarantee, find a counterexample.

[Hint: Note that X_n converges almost surely implies X_n converges in probability. Therefore, if you correctly proved that X_n converges almost surely, then you don't need to prove X_n converges in probability. Similarly, if you correctly proved that X_n does not converge in probability, then you don't need to prove X_n does not converge almost surely.]

(a) $Y_n \overset{\text{i.i.d.}}{\sim} \exp(\lambda)$ for some $\lambda > 0$. $X_n = \min\{Y_1, Y_2, \ldots, Y_n\}$. [8 points]

(b) $Y_n \overset{\text{i.i.d.}}{\sim}$ Uniform$([1, 2])$. $X_n = \left(\prod_{i=1}^{n} Y_i\right)^{1/n}$. [8 points]

(c) X_n satisfies that $\mathbb{E}[(X_n - X)^2] \to 0$ for some random variable X. [8 points]

(a) X_n converges to 0 almost surely, [2 points] and thus in probability. [2 points] Note that $X_n \sim \exp(n\lambda)$. For any $\epsilon > 0$, we define event $A_n = \{|X_n| > \epsilon\}$. Since $\mathrm{P}(A_n) = \exp(-n\lambda\epsilon)$, we have $\sum_{n=1}^{\infty} \mathrm{P}(A_n) < \infty$ [2 points]. By Borel–Cantelli lemma, $\mathrm{P}(|X_n| > \epsilon \text{ i.o.}) = 0$ for any $\epsilon > 0$ [2 points]. Therefore, $X_n \to 0$ a.s.

(b) X_n converges to $\frac{4}{e}$ a.s. [2 points] and thus in probability [2 points]. Let $Z_n = \ln Y_n$. Then $\ln X_n = \frac{1}{n}\sum_{i=1}^{n} \ln Y_i = \frac{1}{n}\sum_{i=1}^{n} Z_i$. [1 point] By SLLN, $\ln X_n \to \mathbb{E}[Z_n]$. [1 point] Since $\mathbb{E}[Z_n] = \mathbb{E}[\ln Y_n] = \int_{x=1}^{2} \ln x \; dx = (x \ln x - x)|_{x=1}^{2} = 2\ln 2 - 1$ [1 point], by continuous mapping, we have $X_n = \exp(\ln X_n) \to \exp(\mathbb{E}[Z_n]) = 4/e$. [1 point]

(c) $X_n \to X$ in probability [2 points]. Note that for any $\epsilon > 0$, $\mathrm{P}(|X_n - X| > \epsilon) = \mathrm{P}((X_n - X)^2 > \epsilon^2) \le \frac{\mathbb{E}[(X_n - X)^2]}{\epsilon^2}$ by Markov Inequality [1 point]. Since $\mathbb{E}[(X_n - X)^2] \to 0$, we have $\mathrm{P}(|X_n - X| > \epsilon) \to 0$ [1 point]. However, X_n does not necessarily converge to anything a.s. [2 points]. As a counter-example, let $X \equiv 0$, and for each $i \ge 0$, we choose k uniformly from $\{2^i, 2^i+1, \ldots, 2^{i+1}-1\}$, and set $X_k = 1$ and other $X_j = 0$, $\forall j \in \{2^i, 2^i+1, \ldots, 2^{i+1}-1\} \backslash \{k\}$. [1 point] By construction, X_n deviates from 0 i.o. Also, $\mathbb{E}[(X_n - X)^2] = \mathrm{P}(X_n = 1) \to 0$. [1 point]

2 EECS126 Map [24 points]

An EECS126 student walks randomly in the map displayed in Figure 1, where at each time $t \in \mathbb{Z}_+$, they transit uniformly at random to an adjacent location directly reachable from the student's current position, as indicated by the directed edge in the map (notice that the BLISS Lab is locked from the inside).

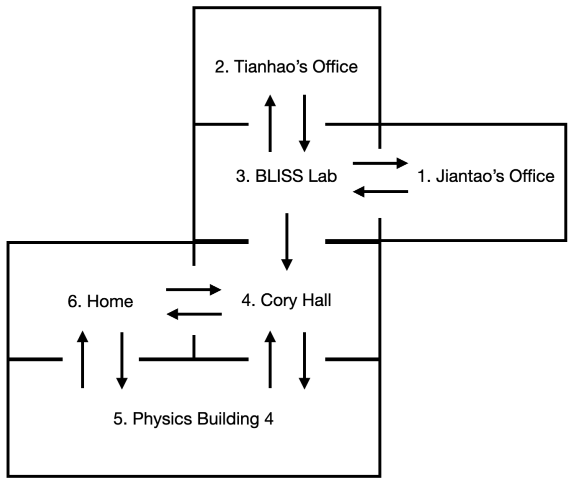

Figure 1: Map

(a) Let the student start from Jiantao's Office at time $t = 0$, what is the probability that the student is in Tianhao's office at time $t = 4$ and then goes to the Cory Hall at time $t = 6$? (4 points)

(b) Find the recurrent and transient classes. (6 points)

(c) Find the stationary distribution over the recurrent class. (4 points)

(d) Let the student start from Jiantao's Office at time $t = 0$, find the expected number of returns to Jiantao's Office (starting at Jiantao's office itself does not count as a return). (6 points)

(e) Find the probability of returning to Jiantao's office in finite time, starting from Jiantao's Office. (4 points)

(a) We label the locations by:

(1) Jiantao's Office
(2) Tianhao's Office
(3) BLISS Lab
(4) Cory Hall
(5) Physics Building 4
(6) Home

Then

$$\begin{aligned}\mathbb{P}(X_4 = 2, X_6 = 4) &= \mathbb{P}(X_4 = 2) \cdot \mathbb{P}(X_6 = 4 | X_4 = 2) \text{ (2 points)} \\ &= 2/9 \cdot 1/3 \\ &= 2/27 \text{ (2 points)}.\end{aligned}$$

(b) Transient class: (1,2,3). (3 points)

Recurrent class: (4,5,6). (3 points)

(c) By solving the balance equation (omitted) (2 points),

$$\pi = (1/3, 1/3, 1/3). \text{ (2 points)}$$

(d) Define $T_i = \mathbb{E}[\text{num of returns to Jiantao's Office}|\text{start at state } i]$ for $i = 1, 2, 3$. By first step equation:

$$T_1 = T_3,\ T_2 = T_3,\ T_3 = (T_1 + 1)/3 + T_2/3. \text{ (4 points)}$$

It follows that $T_1 = 1$. (2 points)

(e) Let τ_1 denote the probability of returning to Jiantao's office from Jiantao's Office. Then one plus the number of returns to Jiantao's Office follows a geometric distribution with rate τ_1. Using the formula for the expectation of geometric distributions:

$$T_1 = \frac{\tau_1}{1 - \tau_1}. \text{ (2 points)}$$

It follows that $\tau_1 = 1/2$. (2 points)

3 Random Walk with Reflection [26 points]

Let $n \in \mathbb{Z}_+$. Consider the unbiased random walk $(X_t)_{t=0,1,2,\ldots}$ over $1, 2, \cdots, 2n$ (with reflection at the boundary $1, 2n$). Specifically, the transition probability is given by

$$\mathbb{P}(X_{t+1} = i+1 | X_t = i) = \mathbb{P}(X_{t+1} = i-1 | X_t = i) = 1/2, \ \forall i = 2, 3, \ldots, 2n-1$$
$$\mathbb{P}(X_{t+1} = 2 | X_t = 1) = \mathbb{P}(X_{t+1} = 2n-1 | X_t = 2n) = 1.$$

(a) Find the stationary distribution. Is this Markov chain reversible? (6 points)

(b) Suppose $X_0 = 2n$. For each of the following sequences of random variables, study its convergence. (8 points)

That is, you are expected to answer: (i) if the sequence converges; (ii) if so, find and prove its limit and specify the strongest mode of convergence it satisfies among {almost surely, in probability, in distribution}; (iii) if not, briefly explain the reason.

- $(X_t)_{t=1,2,\ldots}$
- $(Y_t)_{t=1,2,\ldots}$ where $Y_t = \frac{1}{t}\sum_{j=1}^{t} X_j$

(c) Suppose $X_0 = m$ where $1 < m < 2n$. What is the probability that the walker at $2n$ when it first hits the boundary? (6 points)

Hint: Let p_i denote the probability that given $X_0 = i$, X_t is at $2n$ when it first hits the boundary. What is the relationship between p_i's?

(d) Suppose $X_0 = m$ where $1 < m < 2n$. Let T_b be the first time that the walker hits the boundary. Use the following result to find $\mathbb{E}[T_b]$. (6 points)

Theorem: Let τ be a stopping time with respect to $(X_t)_{t\in\mathbb{N}}$ (for example, you can assume without proof that T_b is a stopping time). If a random process $(M_t)_{t\in\mathbb{N}}$ satisfies

$$\mathbb{E}[M_t | X_1, \ldots, X_{t-1}] = M_{t-1}, \ \forall t \in \mathbb{Z}_+,$$

then $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

Hint: set $M_t = (X_t - m)^2 - t$.

(a) By solving the balance equation (omitted),

$$\pi_i = \begin{cases} 2/(4n-2), & i = 2, 3, \ldots, 2n-1 \\ 1/(4n-2), & i = 1, 2n \end{cases}. \text{ (3 points)}$$

Since it satisfies Detailed Balance Equation, the Markov Chain is reversible. (3 points)

(b) The Markov chain is irreducible. It is further obvious that it is periodic. It follows that

- X_t does not converge (2 points), because we notice that X_t must have the same parity with t (2 points).

- By the big theorem, $Y_t \to \sum_{i=1}^{2n} i \cdot \pi_i = \frac{4n^2-1}{4n-2}$ almost surely. (4 points, -2 points if the convergence notion is wrong.).

(c) Let p_i be as defined in the hint, then by first step analysis

$$p_i = \begin{cases} 1, & i = 2n \\ \frac{1}{2}p_{i+1} + \frac{1}{2}p_{i-1}, & 1 < i < 2n \\ 0, & i = 1 \end{cases}. \text{ (3 points)}$$

Solving this gives $p_i = \frac{i-1}{2n-1}$ (3 points).

(d) On one hand, $\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0$ (1 points).

On the other hand,

$$\begin{aligned} \mathbb{E}[M_\tau] &= \mathbb{E}[(X_\tau - m)^2 - \tau] \\ &= \frac{m-1}{2n-1} \cdot (2n-m)^2 + \frac{2n-m}{2n-1} \cdot (1-m)^2 - \mathbb{E}[\tau] \\ &= (m-1)(2n-m) - \mathbb{E}[\tau] \text{ (2 points).} \end{aligned}$$

Let $\zeta_t = \mathbb{1}(\text{move to the right at } t \text{ th round})$, then

$$\begin{aligned} \mathbb{E}[M_t | X_1, \ldots, X_{t-1}] &= M_{t-1} + 2M_{t-1}\mathbb{E}[\zeta_t | X_1, \ldots, X_{t-1}] + \mathbb{E}[\zeta_t^2 | X_1, \ldots, X_{t-1}] - 1 \\ &= M_{t-1}. \text{ (2 points)} \end{aligned}$$

Then the theorem implies $(m-1)(2n-m) - \mathbb{E}[\tau] = 0$. It follows that $\mathbb{E}[\tau] = (m-1)(2n-m)$ (1 points).

4 Homework Party [25 points]

There is a homework party at Cory 400! Starting from time 0, students (assuming there are sufficiently many students) arrive at Cory 400 according to a Poisson process of rate λ_s. GSIs (assuming there are sufficiently many GSIs) arrive at Cory 400 according to an independent Poisson process of rate λ_g, and the professor (assuming the professor can arrive for multiple times) arrives at Cory 400 according to an independent Poisson process of rate λ_p. Whenever a student or a GSI arrives, they will stay at Cory 400 until the professor arrives. Whenever the professor arrives, the professor will immediately solve all the questions, and all people (students, GSIs, and the professor) will immediately leave Cory 400 (the length of the interval between the professor's arrival and all people leaving can be viewed as 0).

(a) Consider the time interval $[0, T]$ for some $T > 0$. Denote $N_T^{(p)}$ as the number of arrivals of the professor during time interval $[0, T]$. Please find $\mathbb{E}[N_T^{(p)}]$. [5 points]

(b) Let N denote the number of people at Cory 400 (excluding the professor) when the professor arrives at Cory 400 for the second time. What is the probability mass function of N? What is $\mathbb{E}[N]$? [9 points]

(c) Let T_e denote the total time that there is nobody at Cory 400 before the professor's 126th arrival. What is $\mathbb{E}[T_e]$? [5 points]

[Hint: Let T_i denote the total time when there is nobody at Cory 400 between professor's $(i-1)$-th and i-th arrival. What is the relationship between T_i for different i?]

(d) Let T_n denote the total time that there is nobody at Cory 400 before n-th person's arrival. What is $\mathbb{E}[T_n]$? (You may first calculate $\mathbb{E}[T_1]$ and express $\mathbb{E}[T_n]$ in terms of $\mathbb{E}[T_1], \mathbb{E}[T_2], \ldots, \mathbb{E}[T_{n-1}]$ and leave them in your answer) [6 points]

(a) Note that $N_T^{(p)} \sim \text{Poisson}(\lambda_p T)$. [3 points] Therefore, $\mathbb{E}[N_T^{(p)}] = \lambda_p T$. [2 points]

(b) It is equivalent to considering the number of people at the Professor's first arrival. Consider the merge of the Poisson process. People arrive at Cory 400 according to a Poisson process of rate $(\lambda_s + \lambda_g + \lambda_p)$ [2 points], where for each person arriving, the probability that they are the professor is $\frac{\lambda_p}{\lambda_s+\lambda_g+\lambda_p}$. [2 points] Therefore,

$$\mathrm{P}(N = n) = \left(\frac{\lambda_s + \lambda_g}{\lambda_s + \lambda_g + \lambda_p}\right)^n \frac{\lambda_p}{\lambda_s + \lambda_g + \lambda_p} \text{ [3 points]}$$

which is a shifted Geometric distribution, i.e., $N + 1 \sim \text{Geo}(p)$ where $p = \frac{\lambda_p}{\lambda_s+\lambda_g+\lambda_p}$. Therefore, $\mathbb{E}[N] = \frac{1}{p} - 1 = \frac{\lambda_s+\lambda_g}{\lambda_p}$ [2 points].

(c) Let T_i denote the total empty time between the professor's $(i-1)$-th arrival and i-th arrival (the 0-th arrival is time 0). Then $T_e = \sum_{i=1}^{126} T_i$ and all T_i are i.i.d. [2 points] Since $T_1 \sim \exp(\lambda_s + \lambda_g + \lambda_p)$, we have $\mathbb{E}[T_1] = \frac{1}{\lambda_s+\lambda_g+\lambda_p}$ and thus $\mathbb{E}[T_e] = \frac{126}{\lambda_s+\lambda_g+\lambda_p}$. [3 points]

(d) We call the time empty when there is nobody at Cory 400. For convenience, let $p = \frac{\lambda_p}{\lambda_p+\lambda_s+\lambda_g}$. Let i be the index of the professor's first arrival. Then the expected empty time before i-th person's (i.e., the professor's) arrival is $\frac{1}{\lambda_p+\lambda_s+\lambda_g}$. [1 point] Also, the expected empty time between i-th person's arrival and n-th person's arrival is $\mathbb{E}[T_{n-i}]$ for $i < n$. [1 point] In particular, $\mathbb{E}[T_1] = \frac{1}{\lambda_p+\lambda_s+\lambda_g}$. [2 points] Therefore,

$$\begin{aligned}
\mathbb{E}[T_n] &= \left(\sum_{i=1}^{n-1}(1-p)^{i-1}p(\mathbb{E}[T_1]+\mathbb{E}[T_{n-i}])\right) + (1-p)^{n-1}\mathbb{E}[T_1] \\
&= \mathbb{E}[T_1] + \sum_{i=1}^{n-1}(1-p)^{i-1}p\mathbb{E}[T_{n-i}] \\
&= \mathbb{E}[T_1] + \sum_{i=1}^{n-1}\left(\frac{\lambda_s+\lambda_g}{\lambda_s+\lambda_g+\lambda_p}\right)^{i-1}\frac{\lambda_p}{\lambda_s+\lambda_g+\lambda_p}\mathbb{E}[T_{n-i}]. \text{ [2 points]}
\end{aligned}$$