UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 01
Spring 2024

1. **Borel–Cantelli Lemma**

   If $A_1, A_2, \ldots$ is a sequence of events with $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, then

   $$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) = 0.$$

   Remark: later we will see how Borel–Cantelli may be used to show some laws of large numbers.

   Solution: If infinitely many of $A_1, A_2, \ldots$ occur, then at least one of $A_n, A_{n+1}, \ldots$ occurs for any $n \in \mathbb{Z}_{>0}$. So,

   $$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) \le \Pr\left(\bigcup_{m=n}^{\infty} A_m\right) \le \sum_{m=n}^{\infty} \mathbb{P}(A_m) \xrightarrow[n\to\infty]{} 0$$

   because $\sum_{i=1}^{\infty} \mathbb{P}(A_i)$ converges. In more detail,

   $$\sum_{m=n}^{\infty} \mathbb{P}(A_m) = \sum_{m=1}^{\infty} \mathbb{P}(A_m) - \sum_{m=1}^{n-1} \mathbb{P}(A_m),$$

   and as $n \to \infty$, the second term converges to $\sum_{m=1}^{\infty} \mathbb{P}(A_m)$, so $\sum_{m=n}^{\infty} \mathbb{P}(A_m)$ converges to 0 as $n \to \infty$.

   Note: This result is incredibly useful for proving convergence results.

2. **Independence**

Events $A, B \in \mathcal{F}$ are said to be **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

a. Show that if events A, B are independent, then the probability exactly one of the events occurs is

$$\mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A)\,\mathbb{P}(B).$$

b. Show that if the event A is independent of itself, then $\mathbb{P}(A) = 0$ or 1.

Solution:

a. The probability of the event that exactly one of A and B occurs is

$$\begin{aligned}
&\mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B) \\
&= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A)\,\mathbb{P}(B).
\end{aligned}$$

b. $\mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A)$, so $\mathbb{P}(A) = \mathbb{P}(A)^2$. This implies that $\mathbb{P}(A) \in \{0, 1\}$.

3. **Puffcaps**

Consider a deck of 40 cards. Puffcaps are traps which are planted on cards in the deck and activated when drawn. At the beginning of the game, 40 puffcaps are planted on random cards, uniformly at random and independently of each other. A card may have multiple puffcaps planted on it. Given that Axel has drawn 20 cards and already activated 20 puffcaps, what is the probability that the next card he draws activates zero puffcaps?

Solution: Let A be the given event, and let B be the event whose conditional probability we wish to find. By Bayes' rule, this is

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

If we consider puffcaps as balls and cards as bins, we can leverage the uniformity of the sample space $\{1, \ldots, 40\}^{40}$ and find the probabilities as proportions. For instance,

$$\mathbb{P}(A) = \binom{40}{20} 20^{20} (40 - 20)^{40-20} \cdot 40^{-40} = \binom{40}{20} \frac{1}{2^{40}}.$$

If there are 20 puffcaps on 20 cards, then the remaining 20 puffcaps are on the remaining 20 cards. There are $\binom{40}{20}$ choices of which 20 puffcaps belong to the first group, and for each puffcap, there are $20 = 40 - 20$ choices for which card it is planted on. By similar combinatorial reasoning,

$$\mathbb{P}(A \cap B) = \binom{40}{20} (20)^{20} (19)^{40-20} \cdot 40^{-40}.$$

Therefore

$$\mathbb{P}(B \mid A) = \left(\frac{19}{20}\right)^{20},$$

which can be explained as the probability that every one of the 20 remaining puffcaps, out of the 20 possible remaining cards they can be on, all avoid the 1 next card.