UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 02
Spring 2024

1. **Law of the Unconscious Statistician**

a. Prove the *Law of the Unconscious Statistician* (LOTUS): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X\colon \Omega \to \mathbb{Z}$ and $F\colon \mathbb{Z} \to \mathbb{Z}$ be random variables. Note that the composition $Y = F(X)\colon \Omega \to \mathbb{Z}$ is another random variable. If $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and $\mathbb{E}_{\mathcal{L}_X}$ is expectation with respect to the *law* of X on $\mathbb{Z}$, then

$$\mathbb{E}(F(X)) = \mathbb{E}_{\mathcal{L}_X}(F).$$

You should assume that Ω is **discrete** for the sake of simplicity, although LOTUS holds more generally.

b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all sequences of independent fair coin tosses. Formulate N, the minimum number of tosses needed until we see heads, as a random variable on Ω.

c. Find $\mathbb{E}(N^2)$.
Hint: By the linearity of expectation, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. You may use the Law of the Unconscious Statistician from part a, and the following identity:

$$\sum_{k=1}^{\infty} k(k-1)x^{k-2} = \frac{d}{dx}\sum_{k=1}^{\infty} kx^{k-1}.$$

Solution:

a. By the definition of expectation, the left-hand side is equal to

$$\mathbb{E}(F(X)) = \sum_{y\in\mathbb{Z}} y\,\mathbb{P}(F(X) = y)$$
$$= \sum_{y\in\mathbb{Z}} y\,\mathbb{P}(X \in \{x : F(x) = y\}).$$

The $\mathbb{Z}$ above refers to the second $\mathbb{Z}$ in $\Omega \xrightarrow{X} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$. Now, the law $\mathcal{L}_X$ of X is a probability measure on (the first) $\mathbb{Z}$, such that

$$\mathcal{L}_X(B) = \mathbb{P}(X \in B) \quad \text{for } B \subset \mathbb{Z}.$$

So, the above is precisely equal to

$$= \sum_{y\in\mathbb{Z}} y\mathcal{L}_X(\{x : F(x) = y\})$$
$$= \sum_{y\in\mathbb{Z}} y\mathcal{L}_X(F = y) = \mathbb{E}_{\mathcal{L}_X}(F).$$

b. We write each outcome ω as $(\omega_1, \omega_2, \omega_3, \ldots)$, where $\omega_n \in \{H, T\}$ is the result of the nth toss. Then, we define N by

$$N(\omega) = \min\{n \geq 1 : \omega_n = H\}.$$

c. Per the hint, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. We observe that N is a Geometric random variable with parameter $p = \frac{1}{2}$, which has expected value $\mathbb{E}(N) = 2$. Now, by the Law of the Unconscious Statistician,

$$\begin{aligned}\mathbb{E}(N(N-1)) &= \sum_{k=1}^{\infty} k(k-1)\,\mathbb{P}(N = k) \\ &= \sum_{k=1}^{\infty} k(k-1)p(1-p)^{k-1}\end{aligned}$$

Pulling out a factor of $p(1-p)$, we can apply the final hint.

$$\begin{aligned}&= p(1-p)\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2} \\ &= -p(1-p)\frac{d}{dp}\sum_{k=1}^{\infty} k(1-p)^{k-1}\end{aligned}$$

Note that $\mathbb{E}(N) = \frac{1}{p}$ is, by definition, equal to $p\sum_{k=1}^{\infty} k(1-p)^{k-1}$. Then,

$$\begin{aligned}&= -p(1-p)\frac{d}{dp}\frac{1}{p^2} \\ &= \frac{2(1-p)}{p^2}.\end{aligned}$$

For $p = \frac{1}{2}$, we find that $\mathbb{E}(N^2) = 4 + 2 = 6$.

Alternatively, we can observe the following recurrence relation. We remark that the approach above of finding $\mathbb{E}(N(N-1))$ is also applicable when N is *Poisson*, but not the following approach.

$$\begin{aligned}\mathbb{E}(N^2) = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1} &= 1^2 \cdot p + (1-p)\sum_{k=1}^{\infty}(k+1)^2 p(1-p)^{k-1} \\ &= p + (1-p)\,\mathbb{E}((N+1)^2) \\ &= p + (1-p)(\mathbb{E}(N^2) + 2\,\mathbb{E}(N) + 1).\end{aligned}$$

2. **Minimum of Geometrics**

Suppose that you are flipping two coins at the same time. The coins are independent of each other, and have probability of heads p and q respectively. Starting at time step 1, at each time step, you flip both coins, and stop if at least one shows heads. What is the expected number of time steps before you stop (including the last flip)? Use this to prove that the minimum of two Geometric random variables is itself Geometric.

Solution: Let Z be the number of time steps, i.e. number of flips, before you stop. If we let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ be the number of flips before each individual coin shows heads, then $Z = \min(X, Y)$. We find that

$$\mathbb{P}(Z > k) = \mathbb{P}(X > k, Y > k) = \mathbb{P}(X > k) \cdot \mathbb{P}(Y > k) = (1 - p)^k \cdot (1 - q)^k$$

by independence. We note that for a Geometric random variable like X, we have $\mathbb{P}(X > k) = (1 - p)^k$, so

$$\mathbb{P}(Z > k) = \big((1 - p)(1 - q)\big)^k$$

shows that Z is a Geometric random variable with parameter $1 - (1 - p)(1 - q)$. The expected number of time steps is now simply $\mathbb{E}(Z) = \frac{1}{1-(1-p)(1-q)}$.

3. **Variance**

If $X_1, \dots, X_n$, where $n \in \mathbb{Z}_{>0}$, are i.i.d. random variables with zero-mean and unit variance, compute the variance of $(X_1 + \cdots + X_n)^2$. You may leave your answer in terms of $\mathbb{E}[X_1^4]$, which is assumed to be finite.

Solution: For any random variable Y, $\operatorname{var} Y = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$. Applying this to the random variable $Y = X_1 + \cdots + X_n$, we obtain $\mathbb{E}[(X_1 + \cdots + X_n)^2] = \operatorname{var}(X_1 + \cdots + X_n) + \mathbb{E}[X_1 + \cdots + X_n]^2 = \sum_{i=1}^n \operatorname{var} X_i = n$, where in the last step we used linearity of variance because of the assumed independence of the random variables, and also the fact that they are zero-mean. Next we compute $\mathbb{E}[(X_1 + \cdots + X_n)^4] = \mathbb{E}[\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n X_i X_j X_k X_l]$, but notice that for any term in the summation $X_i X_j X_k X_l$, if any of the indices appears exactly once (suppose it is i), then $\mathbb{E}[X_i X_j X_k X_l] = \mathbb{E}[X_i] \, \mathbb{E}[X_j X_k X_l] = 0$ by independence. Hence, the only terms that survive are terms of the form $\mathbb{E}[X_i^4]$ and $\mathbb{E}[X_i^2 X_j^2]$ for distinct indices $i, j \in \{1, \dots, n\}$. There are exactly n terms of the form $\mathbb{E}[X_i^4]$ as i ranges over $1, \dots, n$. To count the second type of term, note that there are $\binom{n}{2}$ ways to choose the two indices i and j, and $\binom{4}{2} = 6$ ways to permute i and j. To illustrate, note that the term $X_1^2 X_2^2$ can arise in one of 6 ways, as $X_1X_1X_2X_2$, $X_1X_2X_1X_2$, $X_1X_2X_2X_1$, $X_2X_1X_1X_2$, $X_2X_1X_2X_1$, and $X_2X_2X_1X_1$. Thus:

$$\begin{aligned}
\mathbb{E}[(X_1 + \cdots + X_n)^4] &= n \, \mathbb{E}[X_1^4] + 6 \binom{n}{2} \mathbb{E}[X_1^2 X_2^2] \\
&= n \, \mathbb{E}[X_1^4] + 3n(n-1) \, \mathbb{E}[X_1^2] \, \mathbb{E}[X_2^2] \\
&= n \, \mathbb{E}[X_1^4] + 3n(n-1).
\end{aligned}$$

So, the variance is

$$\begin{aligned}
\operatorname{var}\big((X_1 + \cdots + X_n)^2\big) &= \mathbb{E}[(X_1 + \cdots + X_n)^4] - \mathbb{E}[(X_1 + \cdots + X_n)^2]^2 \\
&= n \, \mathbb{E}[X_1^4] + 3n(n-1) - n^2.
\end{aligned}$$