UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 02
Spring 2024

1. **Law of the Unconscious Statistician**

   a. Prove the *Law of the Unconscious Statistician* (LOTUS): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X\colon \Omega \to \mathbb{Z}$ and $F\colon \mathbb{Z} \to \mathbb{Z}$ be random variables. Note that the composition $Y = F(X)\colon \Omega \to \mathbb{Z}$ is another random variable. If $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and $\mathbb{E}_{\mathcal{L}_X}$ is expectation with respect to the *law* of X on $\mathbb{Z}$, then

   $$\mathbb{E}(F(X)) = \mathbb{E}_{\mathcal{L}_X}(F).$$

   You should assume that Ω is **discrete** for the sake of simplicity, although LOTUS holds more generally.

   b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all sequences of independent fair coin tosses. Formulate N, the minimum number of tosses needed until we see heads, as a random variable on Ω.

   c. Find $\mathbb{E}(N^2)$.
   Hint: By the linearity of expectation, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. You may use the Law of the Unconscious Statistician from part a, and the following identity:

   $$\sum_{k=1}^{\infty} k(k-1)x^{k-2} = \frac{d}{dx}\sum_{k=1}^{\infty} kx^{k-1}.$$

2. **Minimum of Geometrics**

   Suppose that you are flipping two coins at the same time. The coins are independent of each other, and have probability of heads p and q respectively. Starting at time step 1, at each time step, you flip both coins, and stop if at least one shows heads. What is the expected number of time steps before you stop (including the last flip)? Use this to prove that the minimum of two Geometric random variables is itself Geometric.

3. **Variance**

If $X_1, \ldots, X_n$, where $n \in \mathbb{Z}_{>0}$, are i.i.d. random variables with zero-mean and unit variance, compute the variance of $(X_1 + \cdots + X_n)^2$. You may leave your answer in terms of $\mathbb{E}[X_1^4]$, which is assumed to be finite.