UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 03
Spring 2024

1. **Expected Norm**

   Pick two points $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ independently and uniformly in $[0, 1]^2$. Calculate $\mathbb{E}(\|X - Y\|_2^2)$.

   Solution: We observe that $X_1, X_2, Y_1, Y_2 \sim \text{Uniform}([0, 1])$, and

   $$\mathbb{E}(\|X - Y\|_2^2) = \mathbb{E}((X_1 - Y_1)^2) + \mathbb{E}((X_2 - Y_2)^2).$$

   We can calculate $\mathbb{E}((X_1 - Y_1)^2) = \mathbb{E}(X_1^2) - 2\,\mathbb{E}(X_1)\,\mathbb{E}(Y_1) + \mathbb{E}(Y_1^2) = 2 \cdot \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$. By symmetry, we get that $\mathbb{E}(\|X - Y\|_2^2) = \frac{1}{3}$.

2. **Minimum and Maximum of Exponentials**

Let $\lambda_1, \lambda_2 > 0$, and $X_1 \sim \text{Exponential}(\lambda_1)$, $X_2 \sim \text{Exponential}(\lambda_2)$ are independent. Also, define $U := \min(X_1, X_2)$ and $V := \max(X_1, X_2)$. Show that U and $V - U$ are independent.

Solution: For $u, w > 0$,

$$\begin{aligned}
\Pr(U \le u, V - U \le w, X_1 < X_2) &= \Pr(X_1 \le u, X_1 < X_2 \le X_1 + w) \\
&= \int_0^u \int_{x_1}^{x_1+w} \lambda_2 \exp(-\lambda_2 x_2)\, dx_2\, \lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \int_0^u \big\{\exp(-\lambda_2 x_1) - \exp\big(-\lambda_2(x_1 + w)\big)\big\} \lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \big(1 - \exp(-\lambda_2 w)\big) \int_0^u \lambda_1 \exp\big(-(\lambda_1 + \lambda_2) x_1\big)\, dx_1 \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2} \big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big)\big(1 - \exp(-\lambda_2 w)\big).
\end{aligned}$$

By symmetry, interchanging the roles of 1 and 2 yields

$$\begin{aligned}
&\Pr(U \le u, V - U \le w, X_2 < X_1) \\
&\qquad = \frac{\lambda_2}{\lambda_1 + \lambda_2} \big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big)\big(1 - \exp(-\lambda_1 w)\big).
\end{aligned}$$

Adding these two expressions yields

$$\begin{aligned}
\Pr(U \le u, V - U \le w) &= \big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big) p_w, \qquad \text{where} \\
p_w := \frac{\lambda_1}{\lambda_1 + \lambda_2} &\big(1 - \exp(-\lambda_2 w)\big) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \big(1 - \exp(-\lambda_1 w)\big).
\end{aligned}$$

The joint CDF splits into a product of factors $\Pr(U \le u) \Pr(V - U \le w)$ which proves independence. To interpret the second term, observe that $\lambda_1/(\lambda_1 + \lambda_2)$ is the probability of the event $\{X_1 < X_2\}$; and conditioned on this event, $V - U \sim \text{Exponential}(\lambda_2)$ by the memoryless property.

3. **Poisson Recursion**

Suppose that X is Poisson distributed with parameter λ. Show that $\mathbb{E}(X^n) = \lambda\,\mathbb{E}((X+1)^{n-1})$, and use this to compute $\mathbb{E}(X^3)$.

Solution:

$$\begin{aligned}\mathbb{E}(X^n) &= \sum_{i=0}^{\infty} i^n \frac{\varepsilon^{-\lambda}\lambda^i}{i!} = \sum_{i=1}^{\infty} i^n \frac{\varepsilon^{-\lambda}\lambda^i}{i!} = \sum_{i=1}^{\infty} i^{n-1} \frac{\varepsilon^{-\lambda}\lambda^i}{(i-1)!} \\ &= \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{\varepsilon^{-\lambda}\lambda^{j+1}}{j!} = \lambda\mathbb{E}[(X+1)^{n-1}].\end{aligned}$$

We are able to throw away the $i = 0$ term in the summation because it is 0. Now, $\mathbb{E}(X^3) = \lambda\mathbb{E}[(X+1)^2]$. Since X is Poisson, $\mathbb{E}(X) = \lambda$, $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda$. Plugging in, $\mathbb{E}(X^3) = \lambda(\lambda^2 + 3\lambda + 1)$.