UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 04
Spring 2024

1. **Really Random Binomial**

Consider the random variables $U \sim \text{Uniform}([0,1])$ and $X|U \sim \text{Binomial}(n, U)$, where X is a binomial random variable with a random success probability. Given that $X = k$, we wish to find the conditional distribution of U, $f_{U|X}(u \mid k)$ using the steps below.

a. Write $f_{U|X}(u \mid k)$ in terms of the distributions of X, U, and $X \mid U$ using Bayes' Rule. Plug in any distribution given in the setup.

b. You may realize that the denominator $\mathbb{P}(X = k)$ of your expression above is hard to evaluate. It requires integrating over values of U and iterative integration by parts. Instead, we resort to an approach based on moment generating functions. Write the mgf of X as a summation in terms of $\mathbb{P}(X = k)$. Then, write $\mathbb{P}(X = k)$ as an integral over values of U and exchange the summation and integration. Use the binomial theorem to absorb the summation so we are left with an integral.

c. Carry out the evaluation of the integral. Use the identity $\frac{1-s^{n+1}}{1-s} = \sum_{i=0}^{n} s^i$ to leave your answer as a summation. Does this expression look like the mgf of some discrete random variable, and which one?

d. Conclude the distribution of X is the distribution of the discrete random variable you found above. Use this to find $\mathbb{P}(X = k)$, then find $f_{U|X}(u \mid k)$.

Solution:

a. By Bayes' rule,

$$f_{U|X}(u \mid k) = \frac{p_{X|U}(k \mid u) \cdot f_U(u)}{\mathbb{P}(X = k)}.$$

We know that $p_{X|U}(k \mid u) = \binom{n}{k} u^k (1-u)^{n-k}$ and $f_U(u) = 1$ by the distributions given.

b.

$$\begin{aligned}
M_X(s) = \mathbb{E}(e^{sX}) &= \sum_{k=0}^{n} e^{sk}\, \mathbb{P}(X = k) \\
&= \sum_{k=0}^{n} e^{sk} \int_0^1 p_{X|U}(k \mid u) \cdot f_U(u)\, du \\
&= \sum_{k=0}^{n} e^{sk} \int_0^1 \binom{n}{k} u^k (1-u)^{n-k} \cdot 1\, du \\
&= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} (u \cdot e^s)^k (1-u)^{n-k} \right) du \\
&= \int_0^1 (e^s u + (1-u))^n\, du.
\end{aligned}$$

c. Now we can substitute $v = (e^s u + (1-u))$, which gives $dv = -(1-e^s)\, du$. Changing the limits appropriately,

$$\begin{aligned} M_X(s) &= -\frac{1}{1-e^s}\int_1^s v^n \, dv \\ &= \frac{1-e^{s(n+1)}}{(n+1)(1-e^s)} \\ &= \frac{1}{n+1}\sum_{k=0}^{n} e^{sk}. \end{aligned}$$

We observe that this MGF corresponds to a discrete random variable taking values in $0, \ldots, n$, each with probability $\frac{1}{n+1}$. In other words, X is a discrete uniform distribution over the specified range since mgf uniquely characterizes the distribution.

d. Thus, our final posterior on U is

$$f_{U|X}(u \mid k) = (n+1) \cdot \binom{n}{k} u^k (1-u)^{n-k}.$$

Remark: Here is another way of understanding why X is uniform. Recall that a binomial distribution can be written as the sum of i.i.d. Bernoulli random variables. Next, recognize that if V is uniform on $[0,1]$, the random variable $\mathbb{1}_{V \leq u}$ is Bernoulli with parameter u. The difference is, the number u is itself another uniform random variable, call it U_{n+1}. This means we can write $X = \sum_{i=1}^{n} \mathbb{1}_{U_i \leq U_{n+1}}$, and $X = k$ implies that U_{n+1} occurs in the $(k+1)$-th position overall, which occurs with uniform probability $\frac{1}{n+1}$ since we are dealing with $n+1$ i.i.d. random variables.

2. **Transforms and Independence**

In this problem, we will make use of multivariate moment generating functions, defined for a random vector $X = (X_1, \dots, X_n)$ as

$$M_X(t) = \mathbb{E}(e^{t \cdot X}) = \mathbb{E}(e^{\sum_{i=1}^n t_i X_i})$$

for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. You may assume that MGFs are unique: if $M_X(t) = M_Y(t)$ for all t, then $X \sim Y$.

Consider the random vector $X = (X_1, \dots, X_n)$. Show that $X_1, \dots, X_n$ are independent if and only if for all $t \in \mathbb{R}^n$,

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t_i).$$

Solution: First suppose that the X_i are independent. Then for any $t \in \mathbb{R}^n$, we have

$$M_X(t) = \mathbb{E}(e^{t \cdot X}) = \mathbb{E}\left(\prod_{i=1}^n e^{t_i X_i}\right) = \prod_{i=1}^n \mathbb{E}(e^{t_i X_i}) = \prod_{i=1}^n M_{X_i}(t_i).$$

Conversely, suppose the product identity holds. Let $X_i' \sim X_i$ be copies such that $X_1', \dots, X_n'$ are independent. For the corresponding random vector $X' = (X_1', \dots, X_n')$, we have

$$M_X(t) = \prod_{i=1}^n \mathbb{E}(e^{t_i X_i}) = \prod_{i=1}^n \mathbb{E}(e^{t_i X_i'}) = M_{X'}(t)$$

for all $t \in \mathbb{R}^n$. But by the uniqueness property of MGFs, this implies that $(X_1, \dots, X_n) \stackrel{\mathsf{d}}{=} (X_1', \dots, X_n')$. Hence $X_1, \dots, X_n$ are independent.

3. **Galton–Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}(\xi) = \mu$ and $\mathrm{var}(\xi) = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year.

a. Compute $\mathbb{E}(X_n)$.

b. Compute $\mathrm{var}(X_n|X_{n-1})$. Then, write $\mathrm{var}(X_n)$ in terms of $\mathrm{var}(X_{n-1})$.

Solution:

a. We first note that $X_0 = N$, so $\mathbb{E}(X_0) = N$ and $\mathrm{var}(X_0) = 0$. Then, conditioned on the number of people in the previous year X_{n-1}, we have

$$\begin{aligned}\mathbb{E}(X_n) = \mathbb{E}(\mathbb{E}(X_n \mid X_{n-1})) &= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{X_{n-1}} \xi_i \mid X_{n-1}\right)\right) \\ &= \mathbb{E}\left(X_{n-1}\,\mathbb{E}(\xi)\right) \\ &= \mu\,\mathbb{E}(X_{n-1}).\end{aligned}$$

By recursion, we find that $\mathbb{E}(X_n) = \mu^n N$.

b. As we computed above, $\mathbb{E}(X_n \mid X_{n-1}) = \mu X_{n-1}$. The conditional variance is $\mathrm{var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$. Then, we have

$$\mathrm{var}\, X_n = \mathbb{E}[\sigma^2 X_{n-1}] + \mathrm{var}(\mu X_{n-1}) = \sigma^2 \mu^{n-1} N + \mu^2\, \mathrm{var}\, X_{n-1}.$$

First, suppose that $\mu = 1$. Then, the recurrence simplifies to $\mathrm{var}\, X_n = \sigma^2 N + \mathrm{var}\, X_{n-1}$, which means that the variance increases linearly:

$$\mathrm{var}(X_n) = \sigma^2 N n.$$

For $\mu \neq 1$, the solution to the recurrence is obtained by finding a pattern after a few iterations:

$$\begin{aligned}\mathrm{var}(X_n) &= \sigma^2\mu^{n-1}N + \mu^2\,\mathrm{var}\, X_{n-1} = \sigma^2\mu^{n-1}N + \sigma^2\mu^n N + \mu^4\,\mathrm{var}\, X_{n-2} \\ &= \cdots = \sigma^2\mu^{n-1}N\sum_{k=0}^{n-1}\mu^k = \sigma^2\mu^{n-1}N\frac{1-\mu^n}{1-\mu}\end{aligned}$$

We have used the formula for a finite geometric series.