UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 04
Spring 2024

1. **Really Random Binomial**

Consider the random variables $U \sim \text{Uniform}([0,1])$ and $X|U \sim \text{Binomial}(n, U)$, where X is a binomial random variable with a random success probability. Given that $X = k$, we wish to find the conditional distribution of U, $f_{U|X}(u \mid k)$ using the steps below.

a. Write $f_{U|X}(u \mid k)$ in terms of the distributions of X, U, and $X \mid U$ using Bayes' Rule. Plug in any distribution given in the setup.

b. You may realize that the denominator $\mathbb{P}(X = k)$ of your expression above is hard to evaluate. It requires integrating over values of U and iterative integration by parts. Instead, we resort to an approach based on moment generating functions. Write the mgf of X as a summation in terms of $\mathbb{P}(X = k)$. Then, write $\mathbb{P}(X = k)$ as an integral over values of U and exchange the summation and integration. Use the binomial theorem to absorb the summation so we are left with an integral.

c. Carry out the evaluation of the integral. Use the identity $\frac{1-s^{n+1}}{1-s} = \sum_{i=0}^{n} s^i$ to leave your answer as a summation. Does this expression look like the mgf of some discrete random variable, and which one?

d. Conclude the distribution of X is the distribution of the discrete random variable you found above. Use this to find $\mathbb{P}(X = k)$, then find $f_{U|X}(u \mid k)$.

2. **Transforms and Independence**

In this problem, we will make use of multivariate moment generating functions, defined for a random vector $X = (X_1, \dots, X_n)$ as

$$M_X(t) = \mathbb{E}(e^{t \cdot X}) = \mathbb{E}(e^{\sum_{i=1}^n t_i X_i})$$

for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. You may assume that MGFs are unique: if $M_X(t) = M_Y(t)$ for all t, then $X \sim Y$.

Consider the random vector $X = (X_1, \dots, X_n)$. Show that $X_1, \dots, X_n$ are independent if and only if for all $t \in \mathbb{R}^n$,

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t_i).$$

3. **Galton–Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}(\xi) = \mu$ and $\text{var}(\xi) = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year.

a. Compute $\mathbb{E}(X_n)$.

b. Compute $\text{var}(X_n|X_{n-1})$. Then, write $\text{var}(X_n)$ in terms of $\text{var}(X_{n-1})$.