UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 05
Spring 2024

1. **Exponential Fun**

   a. Let X_1 and X_2 be i.i.d. Exponential random variables with parameter λ. Show that the PDF of $X_1 + X_2$ is, using convolution, given by

$$f_{X_1+X_2}(x) = \lambda^2 x e^{-\lambda x}.$$

   b. Now, for $n \geq 1$, let $X_1, \ldots, X_n$ be i.i.d. Exponential random variables with parameter λ, and let $S_n := X_1 + \cdots + X_n$. The PDF of S_n is given by the n-fold convolution of the Exponential distribution with itself. Show that the PDF of S_n is

$$f_{S_n}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}.$$

   Remark: The distribution of S_n is also called Erlang(k, λ). We will certainly see the Erlang distribution again in the context of Poisson processes.

   c. Using the above result, consider now the random sum $S_N = X_1 + \cdots + X_N$, where N is a Geometric random variable with parameter p. Compute the PDF of S_N.

Solution:

   a. For $x > 0$, the density is

$$\begin{aligned} f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(s) \cdot f_{X_2}(x-s)\, ds &= \int_0^x \lambda e^{-\lambda s} \cdot \lambda e^{-\lambda(x-s)}\, ds \\ &= \lambda^2 e^{-\lambda x} \int_0^x ds \\ &= \lambda^2 x e^{-\lambda x}. \end{aligned}$$

   b. We proceed by induction, where the case of $n = 1$ is trivial. For the inductive step, we find the convolution

$$\begin{aligned} f_{S_n}(x) = \int_{-\infty}^{\infty} f_{S_{n-1}}(s) \cdot f(x-s)\, ds &= \int_0^x \frac{\lambda^{n-1} s^{n-2} e^{-\lambda s}}{(n-2)!} \cdot \lambda e^{-\lambda(x-s)}\, ds \\ &= \frac{\lambda^n e^{-\lambda x}}{(n-2)!} \int_0^x s^{n-2}\, ds \\ &= \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}. \end{aligned}$$

c. By the law of total probability,

$$\begin{aligned} f_{S_N}(x) = \sum_{n=1}^{\infty} f_{S_n}(x) \cdot \mathbb{P}(N = n) &= \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \cdot p(1-p)^{n-1} \\ &= \lambda p e^{-\lambda x} \sum_{n=1}^{\infty} \frac{(\lambda x(1-p))^{n-1}}{(n-1)!} \\ &= \lambda p e^{-\lambda x} e^{\lambda x(1-p)} \\ &= \lambda p e^{-\lambda p x}. \end{aligned}$$

Thus S_N is another Exponential distribution with parameter λp.

Remark: We also find an intuitive explanation for the Geometric sum of Exponentials being Exponential in the context of splitting Poisson processes. In a stream of arrivals, whose interarrival times are i.i.d. Exponential(λ), we independently mark each arrival as "special" with probability p. Then $\sum_{i=1}^{N} X_i$ asks for the time until the first special arrival. As the special arrivals form their own Poisson process with rate λp, this is simply Exponential(λp).

2. **Basic Properties of Jointly Gaussian Random Variables**

   Let $(X_1, \ldots, X_n)$ be a collection of jointly Gaussian random variables with mean vector μ and covariance matrix Σ. Their joint density is given by, for $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}.$$

   a. Show that $X_1, \ldots, X_n$ are independent if and only if they are pairwise uncorrelated.

   b. Show that any linear combination of $X_1, \ldots, X_n$ will also be a Gaussian random variable. *Hint*: Consider using moment-generating functions.

Solution:

a. Independence implies uncorrelatedness in general, so suppose $X_1, \ldots, X_n$ are pairwise uncorrelated, in which case Σ is a diagonal matrix with diagonal entries $\sigma_1^2, \ldots, \sigma_n^2$. Then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n \sigma_i^2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^n \frac{1}{\sigma_i^2}(x_i-\mu_i)^2\right\} \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2\right\} \\ &= \prod_{i=1}^n f_{X_i}(x_i), \end{aligned}$$

so pairwise uncorrelatedness implies the independence of $X_1, \ldots, X_n$ jointly Gaussian.

b. The moment-generating function of a linear combination $Y = a^\top X = \sum_{i=1}^n a_i X_i$ is

$$\begin{aligned} \phi_Y(t) &= \mathbb{E}(\exp\{ta^\top X\}) = \phi_X(ta) \\ &= \exp\left\{ta^\top \mu - \frac{1}{2}t^2 a^\top \Sigma a\right\}. \end{aligned}$$

Therefore Y is Gaussian with distribution $\mathcal{N}(a^\top \mu, a^\top \Sigma a)$.

3. **Exponential Bounds**

Let $X \sim \text{Exponential}(\lambda)$. For $x > \lambda^{-1}$, find bounds on $\mathbb{P}(X \geq x)$ using Markov's inequality, Chebyshev's inequality, and the Chernoff bound. The Chernoff bound is as follows:

$$\mathbb{P}(X \geq x) = \mathbb{P}(e^{sX} \geq e^{sx}) \leq \frac{\mathbb{E}[e^{sX}]}{e^{sx}} = \frac{M_X(s)}{e^{sx}}, \ \forall s > 0,$$

where the inequality holds due to Markov's inequality.

Solution: Since $\mathbb{E}(X) = \lambda^{-1}$, Markov's inequality gives

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x} = \frac{1}{\lambda x},$$

and from $\text{var}(X) = \lambda^{-2}$, Chebyshev's inequality gives

$$\begin{aligned}\mathbb{P}(X \geq x) &= \mathbb{P}(X - \lambda^{-1} \geq x - \lambda^{-1}) \leq \mathbb{P}(|X - \lambda^{-1}| \geq x - \lambda^{-1}) \\ &\leq \frac{\text{var}(X)}{(x - \lambda^{-1})^2} = \frac{1}{(\lambda x - 1)^2}.\end{aligned}$$

By the Chernoff bound, for any $s > 0$,

$$\mathbb{P}(X \geq x) = \mathbb{P}\left(\exp(sX) \geq \exp(sx)\right) \leq \frac{M_X(s)}{\exp(sx)} = \frac{\lambda}{(\lambda - s)\exp(sx)}.$$

We wish to optimize this bound over $s > 0$; we note that it suffices to maximize the denominator $(\lambda - s)\exp(sx)$. Differentiating,

$$-\exp(sx) + x(\lambda - s)\exp(sx) = 0,$$

so $1 = x(\lambda - s)$, that is, $s = \lambda - x^{-1}$. Thus

$$\begin{aligned}\mathbb{P}(X \geq x) &\leq \frac{\lambda}{(\lambda - (\lambda - x^{-1}))\exp((\lambda - x^{-1})x)} = \frac{\lambda}{x^{-1}\exp(\lambda x - 1)} \\ &= \lambda x \exp\left(-(\lambda x - 1)\right).\end{aligned}$$

Observe that the Chernoff bound is the only one which decreases exponentially with x, which is the true behavior: $\mathbb{P}(X \geq x) = \exp(-\lambda x)$.