UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 05

Spring 2024

1. **Exponential Fun**

   a. Let X_1 and X_2 be i.i.d. Exponential random variables with parameter λ. Show that the PDF of $X_1 + X_2$ is, using convolution, given by

   $$f_{X_1+X_2}(x) = \lambda^2 x e^{-\lambda x}.$$

   b. Now, for $n \geq 1$, let $X_1, \dots, X_n$ be i.i.d. Exponential random variables with parameter λ, and let $S_n := X_1 + \cdots + X_n$. The PDF of S_n is given by the n-fold convolution of the Exponential distribution with itself. Show that the PDF of S_n is

   $$f_{S_n}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}.$$

   Remark: The distribution of S_n is also called Erlang(k, λ). We will certainly see the Erlang distribution again in the context of Poisson processes.

   c. Using the above result, consider now the random sum $S_N = X_1 + \cdots + X_N$, where N is a Geometric random variable with parameter p. Compute the PDF of S_N.

2. **Basic Properties of Jointly Gaussian Random Variables**

   Let $(X_1, \ldots, X_n)$ be a collection of jointly Gaussian random variables with mean vector μ and covariance matrix Σ. Their joint density is given by, for $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left\{-\frac{1}{2}(x-\mu)^\mathsf{T}\Sigma^{-1}(x-\mu)\right\}.$$

   a. Show that $X_1, \ldots, X_n$ are independent if and only if they are pairwise uncorrelated.

   b. Show that any linear combination of $X_1, \ldots, X_n$ will also be a Gaussian random variable. *Hint*: Consider using moment-generating functions.

3. **Exponential Bounds**

Let $X \sim \text{Exponential}(\lambda)$. For $x > \lambda^{-1}$, find bounds on $\mathbb{P}(X \geq x)$ using Markov's inequality, Chebyshev's inequality, and the Chernoff bound. The Chernoff bound is as follows:

$$\mathbb{P}(X \geq x) = \mathbb{P}(e^{sX} \geq e^{sx}) \leq \frac{\mathbb{E}[e^{sX}]}{e^{sx}} = \frac{M_X(s)}{e^{sx}}, \ \forall s > 0,$$

where the inequality holds due to Markov's inequality.