UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 06
Spring 2024

1. **The Weak Law of Large Numbers**

   Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common mean μ and MGF M_X. We assume that $M_X(s)$ is finite when $s \in (-d, d)$ for some $d > 0$. Let

   $$\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}.$$

   a. Show that the transform (or MGF) associated with $\bar{X}_n$ satisfies

   $$M_{\bar{X}_n}(s) = M_X(s/n)^n.$$

   b. Suppose that the transform $M_X(s)$ has a first-order Taylor series expansion around $s = 0$ of the form

   $$M_X(s) = a + bs + o(s),$$

   where $o(s)$ is a function that satisfies $\lim_{s \to 0} o(s)/s = 0$. Find a and b in terms of μ.

   c. Show that for all $s \in (-d, d)$,

   $$\lim_{n \to \infty} M_{\bar{X}_n}(s) = e^{\mu s}.$$

   Hint: If $(a_n)_{n \in \mathbb{N}}$ is a sequence of real numbers converging to a, then $\lim_{n \to \infty}(1 + \frac{a_n}{n})^n = e^a$.

   d. Deduce that $\bar{X}_n \xrightarrow{\mathsf{d}} \mu$. Note that the pointwise convergence of MGFs is equivalent to convergence in distribution.

2. **Borel–Cantelli and the Strong Law**

In this problem, we walk through a proof of the strong law (assuming finite 4th moments) that relies only on basic probability. In class we covered the *Borel-Cantelli lemma*, which states that for events $(A_n)_{n=1}^{\infty}$, if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then

$$\mathbb{P}(A_n \text{ i.o.}) = 0,$$

where we define the event $\{A_n \text{ i.o.}\} = \cap_{n \geq 1} \cup_{m \geq n} A_n$ as the event where infinitely many A_n occur.

a. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}\, X_i = 0$ and $\mathbb{E}\, X_i^4 < \infty$ (and so we also have finite second and third moments). Let $S_n = X_1 + \cdots + X_n$, and compute $\mathbb{E}[S_n^4]$. Write your answer in terms of the moments $\mathbb{E}[X_i^2], \mathbb{E}[X_i^3], \mathbb{E}[X_i^4]$.

b. Fix an $\varepsilon > 0$, and use Markov's inequality to show that, for any n,

$$\mathbb{P}(|S_n/n| > \varepsilon) \leq O(n^{-2}).$$

c. Finally, use Borel-Cantelli to conclude that $\mathbb{P}(\lim_{n \to \infty} S_n/n = 0) = 1$. This a weaker (the full theorem assumes only finite first moments) form of the *strong law of large numbers*.

3. **The CLT Implies the WLLN**

a. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Show that if X_n converges in distribution to a constant c, then X_n converges in probability to c.

b. Now let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with mean μ and finite variance σ^2. Show that the CLT implies the WLLN: that is,

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu)\xrightarrow{\mathsf{d}} Z\sim\mathcal{N}(0,1) \implies \frac{1}{n}\sum_{i=1}^{n}X_i\xrightarrow{\mathbb{P}}\mu,$$

where $\xrightarrow{\mathsf{d}}$ is short for "converges in distribution" and $\xrightarrow{\mathbb{P}}$ for "converges in probability."