UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 08
Spring 2024

1. **Markov Chain Practice**

   Consider a Markov chain with three states 0, 1, 2, and suppose its transition probabilities are $P(0,1) = P(0,2) = \frac{1}{2}$, $P(1,0) = P(1,1) = \frac{1}{2}$, $P(2,0) = \frac{2}{3}$, and $P(2,2) = \frac{1}{3}$.

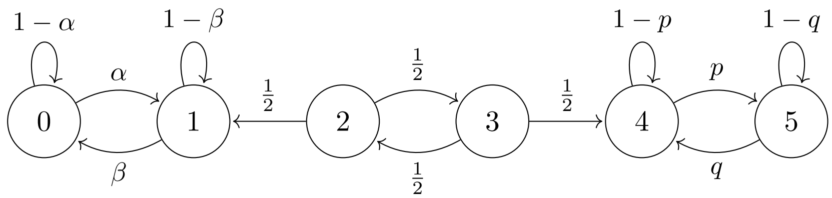
   a. Classify the states in the chain. Is this chain periodic or aperiodic?
   b. In the long run, what fraction of time does the chain spend in state 1?
   c. Suppose that X_0 is chosen according to the steady-state or stationary distribution. What is $\mathbb{P}(X_0 = 0 \mid X_2 = 2)$?

   Solution:

   a. The Markov chain is one recurrent, aperiodic class.
   b. By solving $\pi P = \pi$, we have
   $$\pi = \frac{1}{11}\begin{bmatrix}4 & 4 & 3\end{bmatrix}.$$
   Thus $\pi(1) = 4/11$.
   c. By the definition of conditional probability,
   $$\mathbb{P}(X_0 = 0 \mid X_2 = 2) = \frac{\mathbb{P}(X_0 = 0, X_2 = 2)}{\mathbb{P}(X_2 = 2)} = \frac{\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 2)}{\mathbb{P}(X_2 = 2)}.$$
   Note that we used the fact that the only possible two-step path from $X_0 = 0$ to $X_2 = 2$ in this chain is $0 \to 2 \to 2$. Now, $\mathbb{P}(X_2 = 2) = \mathbb{P}(X_0 = 2)$ because X_0 is chosen according to the stationary distribution π, so
   $$\frac{\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 2)}{\mathbb{P}(X_2 = 2)} = \frac{\pi(0) \cdot (1/2) \cdot (1/3)}{\pi(2)} = \frac{2}{9}.$$

2. **Reducible Markov Chain**

Consider the following Markov chain, where $\alpha, \beta, p, q \in (0,1)$.



a. Find all the recurrent and transient classes.
b. Given that we start in state 2, what is the probability we reach state 0 before state 5?
c. What are all of the possible stationary distributions of this chain?
Hint: Consider the recurrent classes.
d. Suppose we start with initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0,1]$. Does the distribution of the chain converge, and if so, to what?

Solution:

a. The classes are $\{0,1\}$ (recurrent), $\{2,3\}$ (transient), and $\{4,5\}$ (recurrent).
b. Let T_0 and T_5 denote the time it takes to reach states 0 and 5 respectively. Note that exactly one of T_0 and T_5 will be finite. We can set up first-step equations to compute the hitting probability $\mathbb{P}_2(T_0 < T_5) = \mathbb{P}(T_0 < T_5 \mid X_0 = 2)$:

$$\mathbb{P}_2(T_0 < T_5) = \frac{1}{2} + \frac{1}{2}\mathbb{P}_3(T_0 < T_5)$$

$$\mathbb{P}_3(T_0 < T_5) = \frac{1}{2}\mathbb{P}_2(T_0 < T_5).$$

From this, we find that $\mathbb{P}_2(T_0 < T_5) = \frac{2}{3}$.
c. No transient state can support nonzero probability mass at stationarity, so any stationary distribution must be supported on the states $\{0,1,4,5\}$. Now, if we restrict our attention to only the states $\{0,1\}$, we have an irreducible chain with stationary distribution

$$\pi_1 := \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \end{bmatrix}.$$

Similarly, the states $\{4,5\}$ form an irreducible chain with stationary distribution

$$\pi_2 := \frac{1}{p+q}\begin{bmatrix} q & p \end{bmatrix}.$$

Any stationary distribution for the entire chain will be a convex combination of π_1 and π_2, depending on the total amount of stationary mass in each recurrent class. Explicitly, the stationary distributions are of the form

$$\pi = \begin{bmatrix} c\frac{\beta}{\alpha+\beta} & c\frac{\alpha}{\alpha+\beta} & 0 & 0 & (1-c)\frac{q}{p+q} & (1-c)\frac{p}{p+q} \end{bmatrix}$$

for $c \in [0,1]$.

d. The distribution will indeed converge, even without irreducibility. Intuitively, probability mass gradually leaves the transient states $\{2, 3\}$ until eventually, all of the probability mass is supported on the recurrent states. The two recurrent classes then each settle into their own comfortable equilibrium.

Let us use the previous parts to find the limiting distribution. We start in state 2 with probability γ, and we end up in the recurrent class $\{0, 1\}$ with further probability $\frac{2}{3}$. By symmetry, the probability that we end in $\{0, 1\}$ starting from state 3 is $\frac{1}{3}$. Thus, the total probability mass which settles into $\{0, 1\}$ is

$$\frac{2\gamma}{3} + \frac{1-\gamma}{3} = \frac{1}{3} + \frac{\gamma}{3},$$

and the probability mass which settles in $\{4, 5\}$ is $\frac{2}{3} - \frac{\gamma}{3}$. Therefore the chain converges to the stationary distribution found in part c with parameter $c = \frac{1}{3} + \frac{\gamma}{3}$.

3. **Checking Reversibility**

   a. *Cut property.* A **cut** of a graph is a partition of its states S into two disjoint subsets T, $S \setminus T$. Show that for an irreducible Markov chain at stationarity, flow-in equals flow-out holds across any cut of the Markov chain. That is, for any time n,

$$\mathbb{P}(X_n \in T,\, X_{n+1} \in S \setminus T) = \mathbb{P}(X_n \in S \setminus T,\, X_{n+1} \in T).$$

   b. *Sufficient condition for reversibility.* We can convert the transition diagram of any chain into an undirected graph by removing any self-loops and making all edges undirected. For an irreducible chain whose resulting graph is a **tree**, show that if it has a stationary distribution, then it must also satisfy detailed balance.

   (In particular, this shows that positive recurrent birth-death chains are reversible, even on infinite state spaces.)

Solution:

a. Let π denote the stationary distribution of the chain. If T and U are subsets of the state space S, let us write for convenience

$$\mathsf{flow}(T, U) := \sum_{i \in T} \sum_{j \in U} \pi(i) \cdot p(i,j) = \mathbb{P}(X_n \in T,\, X_{n+1} \in U).$$

By stationarity, or the global balance equations, we know that $\mathsf{flow}(T, S) = \mathsf{flow}(S, T)$. Then, we observe that

$$\begin{aligned} \mathsf{flow}(T, S \setminus T) &= \mathsf{flow}(T, S) - \mathsf{flow}(T, T) \\ &= \mathsf{flow}(S, T) - \mathsf{flow}(T, T) = \mathsf{flow}(S \setminus T, T). \end{aligned}$$

But this is precisely the statement that flow-in equals flow-out across the cut $(T, S \setminus T)$.

b. For every edge in a **tree**, there exists a cut crossing only that edge. If a chain is treelike, then every pair of states i, j has a cut that crosses only (i, j). By part a, if the chain is also irreducible (true by assumption) and at stationarity, then the cut property becomes

$$\pi(i) \cdot p(i,j) = \pi(j) \cdot p(j,i) \quad \text{for all } i, j \in S,$$

which is precisely detailed balance, or reversibility.