UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 08
Spring 2024

1. **Markov Chain Practice**

   Consider a Markov chain with three states 0, 1, 2, and suppose its transition probabilities are $P(0,1) = P(0,2) = \frac{1}{2}$, $P(1,0) = P(1,1) = \frac{1}{2}$, $P(2,0) = \frac{2}{3}$, and $P(2,2) = \frac{1}{3}$.

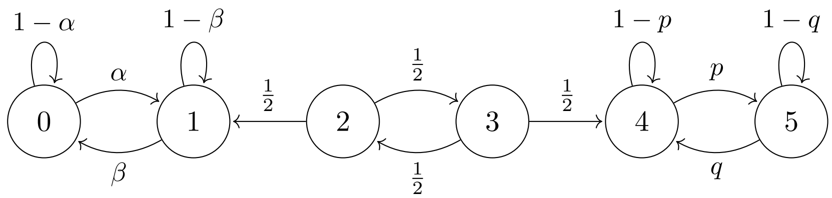
   a. Classify the states in the chain. Is this chain periodic or aperiodic?

   b. In the long run, what fraction of time does the chain spend in state 1?

   c. Suppose that X_0 is chosen according to the steady-state or stationary distribution. What is $\mathbb{P}(X_0 = 0 \mid X_2 = 2)$?

2. **Reducible Markov Chain**

Consider the following Markov chain, where $\alpha, \beta, p, q \in (0, 1)$.



a. Find all the recurrent and transient classes.

b. Given that we start in state 2, what is the probability we reach state 0 before state 5?

c. What are all of the possible stationary distributions of this chain?
Hint: Consider the recurrent classes.

d. Suppose we start with initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?

3. **Checking Reversibility**

a. *Cut property.* A **cut** of a graph is a partition of its states S into two disjoint subsets T, $S \setminus T$. Show that for an irreducible Markov chain at stationarity, flow-in equals flow-out holds across any cut of the Markov chain. That is, for any time n,

$$\mathbb{P}(X_n \in T,\, X_{n+1} \in S \setminus T) = \mathbb{P}(X_n \in S \setminus T,\, X_{n+1} \in T).$$

b. *Sufficient condition for reversibility.* We can convert the transition diagram of any chain into an undirected graph by removing any self-loops and making all edges undirected. For an irreducible chain whose resulting graph is a **tree**, show that if it has a stationary distribution, then it must also satisfy detailed balance.

(In particular, this shows that positive recurrent birth-death chains are reversible, even on infinite state spaces.)