UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 09
Spring 2024

1. **Poisson Process Warmup**

   Give an interpretation of the following fact in terms of a Poisson process with rate λ. If N is Geometric with parameter p and $(X_k)_{k\in\mathbb{N}}$ are i.i.d. Exponential(λ), then $X_1 + \cdots + X_N$ has an Exponential distribution with parameter λp.

   Solution: Consider a Poisson process with rate λ, and split the process by keeping each arrival independently with probability p. In the original process, the interarrival times are i.i.d. Exponential(λ), and $X_1 + \cdots + X_N$ represents the amount of time until the first arrival we keep. By Poisson splitting, we know that the split process is a Poisson process with rate λp, so the time until its first arrival is an Exponential random variable with parameter λp.

2. **Illegal U-Turns**

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal, and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for time $\tau > 0$. Let N be the number of police cars you see before you make a U-turn.

a. Find $\mathbb{E}(N)$.

b. Let $n \geq 2$. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

c. Find the expected time that you wait until you make a U-turn.

Solution:

a. We note that N is equal to the number of successive interarrival intervals that are smaller than τ, where these intervals are independent and each shorter than τ with probability $1 - e^{-\lambda\tau} := 1 - p$. Thus

$$\mathbb{P}(N = k) = e^{-\lambda\tau}(1 - e^{-\lambda\tau})^k = p(1-p)^k,$$

so N is a shifted Geometric random variable with parameter p, i.e. $N+1 \sim \text{Geometric}(p)$, and $\mathbb{E}(N) = \frac{1}{p} - 1 = e^{\lambda\tau} - 1$.

b. Let S_n be the nth interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and n is at most τ, so we want to compute

$$\mathbb{E}(S_n \mid S_n < \tau) = \frac{\int_0^\tau t\lambda e^{-\lambda t}\,dt}{\int_0^\tau \lambda e^{-\lambda t}\,dt}.$$

Using integration by parts in the numerator, we find that the answer is

$$= \frac{\lambda^{-1} - (\tau + \lambda^{-1})e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

c. You make the U-turn at time $T = S_1 + \cdots + S_N + \tau$, with $S_i \leq \tau$ for $i = 1, \ldots, N$, so

$$\begin{aligned}
\mathbb{E}(T) &= \tau + \mathbb{E}(S_1 + \cdots + S_N) \\
&= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n) \cdot \mathbb{E}(S_1 + \cdots + S_N \mid N = n) \\
&= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n) \cdot n \cdot \mathbb{E}(S_1 \mid S_1 \leq \tau) \\
&= \tau + (e^{\lambda\tau} - 1) \cdot \frac{\lambda^{-1} - (\tau + \lambda^{-1})e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.
\end{aligned}$$

3. **Bus Arrivals at Cory Hall**

Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

a. Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. Here, x is a given number, not a random quantity.

b. Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

Solution:

a. The student arrival process is independent of the bus arrival process, so the number of students arrivals in this time interval of length x is Poisson with parameter μx.

b. Let us consider the merged process of student and bus arrivals, which has rate $\lambda + \mu$. Each arrival for the combined process is a bus with probability $p := \frac{\lambda}{\lambda+\mu}$ and a student with probability $\frac{\mu}{\lambda+\mu}$, and these "choices" can be treated as i.i.d. Bernoulli trials. Thus, starting right after the arrival at 9:30 AM, the number of combined arrivals until we see a bus arrival for the first time is Geometric with parameter p. If N is the number of students entering the next bus after 9:30 AM, then for $n \in \mathbb{N}$,

$$\mathbb{P}(N = n) = \left(\frac{\mu}{\lambda + \mu}\right)^n \frac{\lambda}{\lambda + \mu}.$$

Alternate solution. Let $T \sim \text{Exponential}(\lambda)$ be the interarrival time between the 9:30 AM bus arrival and the next bus, and let N be the number of students who arrived between 9:30 AM and 9:30 AM $+ T$. We know that $N \mid T = t \sim \text{Poisson}(\mu t)$, so by the law of total probability,

$$\begin{aligned}
\mathbb{P}(N = n) &= \int_0^\infty \mathbb{P}(N = n \mid T = t) \cdot f_T(t)\, dt \\
&= \int_0^\infty \frac{(\mu t)^n}{n!} e^{-\mu t} \cdot \lambda e^{-\lambda t}\, dt \\
&= \frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \int_0^\infty t^n (\lambda + \mu) e^{-(\lambda+\mu)t}\, dt \\
&= \frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \mathbb{E}(\text{Exponential}(\lambda + \mu)^n) \\
&= \frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \frac{n!}{(\lambda + \mu)^n},
\end{aligned}$$

which simplifies to the same answer as above.