UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 11
Spring 2024

1. **Reversibility of CTMCs**

   We say that a CTMC with transition rate matrix Q and distribution π is *reversible* if π and Q satisfy the detailed balance equations

   $$\pi(i) \cdot q(i, j) = \pi(j) \cdot q(j, i) \quad \forall i, j \in S.$$

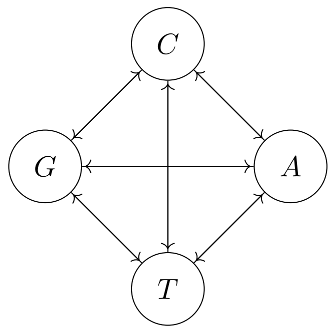
   Show that if π is a reversible distribution for a CTMC, then π is also a stationary distribution for the chain, and moreover the embedded jump chain is also reversible. *Remark*: the converse is also true — the CTMC is reversible if and only if the embedded chain is reversible.

2. **Jukes–Cantor Model**

In this question, we consider a CTMC model for the evolution of DNA over time. Consider a CTMC $(X_t)_{t\geq 0}$ on the states $\mathcal{X} := \{A, C, G, T\}$ with transition rate matrix

$$Q = \begin{bmatrix} -3\lambda & \lambda & \lambda & \lambda \\ \lambda & -3\lambda & \lambda & \lambda \\ \lambda & \lambda & -3\lambda & \lambda \\ \lambda & \lambda & \lambda & -3\lambda \end{bmatrix},$$

where $\lambda > 0$. That is, all edges in the following transition diagram have rate λ:



For $x, y \in \mathcal{X}$, what is $P_t(x, y) := \mathbb{P}(X_t = y \mid X_0 = x)$? What happens as $t \to \infty$?

Hint: consider adding a self-loop with rate λ to each of the four states. Does this change the behavior of the CTMC? Then, condition on whether a transition occurs or not.

3. **Bayesian Estimation of Exponential Distribution**

We have seen the MLE (non-Bayesian perspective) and MAP estimation (Bayesian perspective). In this problem, we will introduce the fully Bayesian approach to statistical estimation.

Suppose that X is Exponential with unknown rate Λ. As a Bayesian practitioner, you have a prior belief that the random variable Λ is equally likely to be λ_1 or λ_2.

Now, you collect one sample X_1 from X.

a. Find the posterior distribution $\mathbb{P}(\Lambda = \lambda_1 \mid X_1 = x_1)$.

b. If we were using the MLE or MAP rule, we would choose a single value λ for Λ, sometimes called a *point estimate*. This amounts to saying X has Exponential distribution with rate λ. In the Bayesian approach, we will instead keep the full information of the posterior distribution of Λ, and we compute the distribution of X as

$$f_X(x) = \sum_{\lambda \in \{\lambda_1, \lambda_2\}} f_{X|\Lambda}(x \mid \lambda) \cdot \mathbb{P}(\Lambda = \lambda \mid X_1 = x_1).$$

Note that we do not necessarily have an Exponential distribution for X anymore. Compute $f_X(x)$ in closed form.

c. From the previous part, you may have guessed that the fully Bayesian approach is often computationally intractable, which is one of the main reasons why point estimates are common in practice. Supposing that $\lambda_1 > \lambda_2$, compute the MAP estimate for Λ, and calculate $f_X(x)$ again using the MAP rule.