UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 01
Spring 2024

1. **Properties of Probability Measures**

   Use the axioms of probability to show the following facts. Clearly identify which axioms are used, and where. You may also use facts from lecture, but identify where you use them.

   a. *Subadditivity*, also known as the "union bound." If $A_1, A_2, \ldots \in \mathcal{F}$, then

   $$\mathbb{P}\left(\bigcup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1} \mathbb{P}(A_i).$$

   b. *Continuity from below.* If $A_1 \subset A_2 \subset \cdots \in \mathcal{F}$, then

   $$\mathbb{P}\left(\bigcup_{i\geq 1} A_i\right) = \lim_{i\to\infty} \mathbb{P}(A_i).$$

   Here and throughout the course, the notation $A \subset B$ means that A is a subset of B (not necessarily a proper subset).

   Hint: If $a_i \geq 0$ for each $i \geq 1$, then we have *monotone convergence*: $\sum_{i\geq 1} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

   Solution:

   a. First, note that the set $A = \bigcup_{i\geq 1} A_i$ is an event, since it is a countable union of events (this is the σ-algebra property of $\mathcal{F}$). Now, decompose A into the disjoint union

   $$A = \bigsqcup_{n\geq 1} (A_n \setminus B_{n-1}),$$

   where $B_n \coloneqq \bigcup_{i=1}^{n} A_i$ (with convention $B_0 \coloneqq \varnothing$). Then, by σ-additivity and monotonicity (since $(A_n \setminus B_{n-1}) \subset A_n$),

   $$\mathbb{P}(A) = \mathbb{P}\left(\bigsqcup_{n\geq 1} (A_n \setminus B_{n-1})\right) = \sum_{n\geq 1} \mathbb{P}(A_n \setminus B_{n-1}) \leq \sum_{n\geq 1} P(A_n).$$

   b. Using the same decomposition as in the previous part, we have $A = \bigsqcup_{i\geq 1}(A_i \setminus A_{i-1})$, and therefore by σ-additivity (twice), and non-negativity of probabilities with monotone convergence,

   $$\mathbb{P}(A) = \sum_{i\geq 1} \mathbb{P}(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

2. **Joint Occurrence**

You know that at least one of the events A_i, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. n is an integer ≥ 2. Show that if the probability of occurrence of any single event is p, and the probability of joint occurrence of any two distinct events is q, we have $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n(n-1)}$.

Solution: By the union bound, since

$$1 = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) = np,$$

we see that $p \geq \frac{1}{n}$. Now we observe that the events $A_i \cap A_j$ for $i < j$, $i, j \in \{1, \ldots, n\}$, are pairwise disjoint, so by finite additivity,

$$1 \geq \mathbb{P}\left(\bigcup_{i<j} A_i \cap A_j\right) = \sum_{i<j} \mathbb{P}(A_i \cap A_j) = \binom{n}{2} q,$$

so $q \leq \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$.

3. **Suspicious Game**

You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

What is the probability that you win the game if you go first?

Solution: Let p denote the probability of randomly selecting a bullet card; p stays the same since you never run out of cards. Because the game ends when the first bullet card is drawn, the number of turns N before the game ends is a Geometric random variable with parameter p. The probability that you win is the probability that N is even, so we have

$$\begin{aligned}\mathbb{P}(\text{win} \mid p) = \mathbb{P}(N \text{ is even} \mid p) &= \sum_{\substack{k=1 \\ k \text{ is even}}}^{\infty} p(1-p)^{k-1} = p(1-p)\sum_{i=0}^{\infty}(1-p)^{2i} \\ &= \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.\end{aligned}$$

Now, we don't know whether $p = \frac{1}{3}$ or $\frac{2}{3}$, so we can use the law of total probability:

$$\begin{aligned}\mathbb{P}(\text{win}) &= \mathbb{P}\left(\text{win} \mid p = \frac{1}{3}\right) \cdot \mathbb{P}\left(p = \frac{1}{3}\right) + \mathbb{P}\left(\text{win} \mid p = \frac{2}{3}\right) \cdot \mathbb{P}\left(p = \frac{2}{3}\right) \\ &= \frac{2}{5} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} \\ &= 0.3625.\end{aligned}$$

Note. It may be tempting to first calculate p as

$$\mathbb{P}(B) = \mathbb{P}(B \mid L) \cdot \mathbb{P}(L) + \mathbb{P}(B \mid L^c) \cdot \mathbb{P}(L^c) = \frac{5}{12},$$

where B is the event of drawing a bullet card and L is the event that your friend is lying. Then, one would plug in $p = \frac{5}{12}$ to find the probability of winning as $\frac{7}{19} \approx 0.3684$. However, this does not work as it is not the case that N is a Geometric random variable with parameter $\frac{5}{12}$: the order of conditioning is reversed in this case.

Remark. The reason why the game is suspicious is because $\frac{1-p}{2-p} \leq \frac{1}{2}$ for $p \in [0, 1]$, so your chances of winning are always unfavorable!

4. **Upperclassmen**

You meet two students in the library. Assume that each student is an upperclassman or underclassman with equal probability, and each student takes EECS 126 with probability $\frac{1}{10}$, independent of each other and independent of their class standing. What is the probability that both students are upperclassmen, given at least one of them is an upperclassman currently taking EECS 126?

Solution: We define the following events:

- A is the given event that at least one student is an upperclassman taking EECS 126.
- B is the event that both students are upperclassmen.
- U is the event that at least one student is an upperclassman.
- E is the event that at least one student is taking EECS 126.

We wish to find $\mathbb{P}(B \mid A)$, which we can rewrite by Bayes' rule as

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)}.$$

To find $\mathbb{P}(A \mid B)$, we notice that given B (both students are upperclassmen), the event A can be reduced to the event E that at least one student is taking EECS 126. The condition that at least one student is an upperclassman becomes redundant. Then, as taking EECS 126 is independent of being an upperclassman,

$$\mathbb{P}(A \mid B) = \mathbb{P}(E \mid B) = \mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \left(\frac{9}{10}\right)^2.$$

The probability of B is straightforward by independence:

$$\mathbb{P}(B) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Now, to find $\mathbb{P}(A)$, we observe that A^c is the event that none of the two students is an upperclassman taking EECS 126. By the given independences,

$$\begin{aligned}\mathbb{P}(A) = 1 - \mathbb{P}(A^c) &= 1 - (\mathbb{P}(\text{is not an upperclassman taking EECS 126}))^2 \\ &= 1 - \left(1 - \frac{1}{2} \cdot \frac{1}{10}\right)^2.\end{aligned}$$

Simplifying our final expression, we get

$$\mathbb{P}(B \mid A) = \frac{\left(1 - \frac{9}{10}\right)^2 \cdot \frac{1}{4}}{1 - \left(\frac{19}{20}\right)^2} = \frac{\frac{19}{400}}{\frac{39}{400}} = \frac{19}{39}.$$

5. **Ascending ball sequence**

Consider a scenario where we have a set of N balls, each uniquely numbered from 1 to N, where N is a positive integer no less than n. We perform a sequence of n independent draws from this set, with each draw consisting of selecting one ball at random and then putting it back before the next draw. What the probability that the sequence of numbers drawn is strictly increasing with each draw?

Solution: The number of possible sequences is N^n. To form an strictly ascending sequence, the n numbers drawn must be distinct. Therefore, the number of ascending sequences is $\binom{N}{n}$. Putting N^n in the denominator and $\binom{N}{n}$ in the numerator, we find that the probability is $\frac{\binom{N}{n}}{N^n}$.

6. **Geometric probability**

Alice and Bob have agreed to meet at Physics Building 4 with the understanding that both will arrive at a random time independently of each other between 7:00 pm and 8:00 pm. Assume that their arrival times are uniformly distributed over this one-hour interval. Calculate the probability that the first person to arrive will have to wait longer than 30 minutes for the second person to arrive.

Solution: The shaded region in the figure represents the event that the first person to arrive will have to wait longer than 30 minutes for the second person to arrive.

To find the probability, we divide the area of the shaded region by the area of the square, which results in 1/4.