UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 01
Spring 2024

1. **Properties of Probability Measures**

   Use the axioms of probability to show the following facts. Clearly identify which axioms are used, and where. You may also use facts from lecture, but identify where you use them.

   a. *Subadditivity*, also known as the "union bound." If $A_1, A_2, \ldots \in \mathcal{F}$, then

   $$\mathbb{P}\left(\bigcup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1} \mathbb{P}(A_i).$$

   b. *Continuity from below.* If $A_1 \subset A_2 \subset \cdots \in \mathcal{F}$, then

   $$\mathbb{P}\left(\bigcup_{i\geq 1} A_i\right) = \lim_{i\to\infty} \mathbb{P}(A_i).$$

   Here and throughout the course, the notation $A \subset B$ means that A is a subset of B (not necessarily a proper subset).

   Hint: If $a_i \geq 0$ for each $i \geq 1$, then we have *monotone convergence*: $\sum_{i\geq 1} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

2. **Joint Occurrence**

   You know that at least one of the events A_i, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. n is an integer ≥ 2. Show that if the probability of occurrence of any single event is p, and the probability of joint occurrence of any two distinct events is q, we have $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n(n-1)}$.

3. **Suspicious Game**

You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

What is the probability that you win the game if you go first?

4. **Upperclassmen**

You meet two students in the library. Assume that each student is an upperclassman or underclassman with equal probability, and each student takes EECS 126 with probability $\frac{1}{10}$, independent of each other and independent of their class standing. What is the probability that both students are upperclassmen, given at least one of them is an upperclassman currently taking EECS 126?

5. **Ascending ball sequence**

Consider a scenario where we have a set of N balls, each uniquely numbered from 1 to N, where N is a positive integer no less than n. We perform a sequence of n independent draws from this set, with each draw consisting of selecting one ball at random and then putting it back before the next draw. What the probability that the sequence of numbers drawn is strictly increasing with each draw?

6. **Geometric probability**

   Alice and Bob have agreed to meet at Physics Building 4 with the understanding that both will arrive at a random time independently of each other between 7:00 pm and 8:00 pm. Assume that their arrival times are uniformly distributed over this one-hour interval. Calculate the probability that the first person to arrive will have to wait longer than 30 minutes for the second person to arrive.