UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 09
Spring 2024

1. **System Shocks**

   For a positive integer n, let $X_1, \ldots, X_n$ be independent Exponentially distributed random variables, each with mean 1. Let $\gamma > 0$. A system experiences shocks at times $k = 1, \ldots, n$, and the size of the shock at time k is X_k.

   a. Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

   b. Suppose instead that the effect of the shocks is cumulative, i.e. the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

   Solution:

   a. The system fails if $\max\{X_1, \ldots, X_n\} > \gamma$, so

$$\begin{aligned}\mathbb{P}(\max\{X_1, \ldots, X_n\} > \gamma) &= 1 - \mathbb{P}(\max\{X_1, \ldots, X_n\} \le \gamma) \\ &= 1 - \prod_{k=1}^{n} \mathbb{P}(X_k \le \gamma) = 1 - (1 - e^{-\gamma})^n.\end{aligned}$$

   b. $\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \mathbb{P}(N_\gamma < n)$, where $(N_t)_{t \ge 0}$ is a Poisson process with rate 1, so

$$\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \sum_{k=0}^{n-1} \frac{\gamma^k}{k!} e^{-\gamma}.$$

2. **Random Telegraph Wave**

Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ, let X_0 be a Bernoulli random variable independent of $(N_t)_{t\geq 0}$, and define $X_t = X_0(-1)^{N_t}$.

a. Does the process $(X_t)_{t\geq 0}$ have independent increments?

b. Calculate $\mathbb{P}(X_t = 1)$ if $\mathbb{P}(X_0 = 1) = p$.

c. Assume that $p = \frac{1}{2}$. Calculate $\mathbb{E}(X_{t+s}X_s)$ for $s, t \geq 0$.

Solution:

a. No, the process does not have independent increments. If it did, for any $0 < t_0 < t_1 < t_2$, we should have $X_{t_2} - X_{t_1}$ is independent of $X_{t_1} - X_{t_0}$. However, suppose $X_0 = 1$ and $X_{t_1} - X_{t_0} = 2$. This means that from t_0 to t_1, X_t increases from -1 to 1. Then it is impossible to have $X_{t_2} - X_{t_1} = 2$, since $X_t \in \{-1, 1\}$ for all $t > 0$ when $X_0 = 1$.

b. Considering the parity of N_t, we observe that

$$\mathbb{P}(X_t = 1) = \mathbb{P}(X_0 = 1 \text{ and } N_t \text{ is even}) = p \cdot \mathbb{P}(N_t \text{ is even}).$$

As $N_t \sim \text{Poisson}(\lambda t)$, the probability that N_t is even is

$$\begin{aligned}\sum_{k \text{ even}} \frac{(\lambda t)^k}{k!} e^{-\lambda t} &= \frac{e^{-\lambda t}}{2}\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda t)^k}{k!}\right) \\ &= \frac{e^{-\lambda t}}{2}(e^{\lambda t} + e^{-\lambda t}) = \frac{1 + e^{-2\lambda t}}{2}.\end{aligned}$$

Note that $(\lambda t)^k = \frac{1}{2}((\lambda t)^k + (-\lambda t)^k)$ for all k, even or odd. Thus $\mathbb{P}(X_t = 1) = \frac{p(1+e^{-2\lambda t})}{2}$.

c. If $X_0 = 0$, then $X_{t+s}X_s = 0$ for all $s, t \geq 0$. For $X_0 = 1$, we observe that

$$\mathbb{P}(X_{t+s}X_s = 1) = \mathbb{P}(N_{t+s} - N_s \text{ is even}) = \mathbb{P}(N_t \text{ is even}) = \frac{1 + e^{-2\lambda t}}{2}$$

and $\mathbb{P}(X_{t+s}X_s = -1) = \frac{1-e^{-2\lambda t}}{2}$. Therefore, we get

$$\begin{aligned}\mathbb{E}(X_{t+s}X_s) &= \mathbb{P}(X_0 = 1) \cdot \mathbb{E}(X_{t+s}X_s \mid X_0 = 1) \\ &= \frac{1}{2}\left[\frac{1 + e^{-2\lambda t}}{2} \cdot 1 + \frac{1 - e^{-2\lambda t}}{2} \cdot -1\right] = \frac{1}{2}e^{-2\lambda t}.\end{aligned}$$

3. **Poisson Process Practice**

Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ. Let T_k, $k \geq 1$ denote the time of the kth arrival. Given $0 \leq s < t$, we write $N(s,t) := N(t) - N(s)$. Compute the following:

a. $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

b. $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

c. $\mathbb{E}(T_2 \mid N(2) = 1)$.

Solution:

a. The event $\{N(1) + N(2,4) + N(3,5) = 0\}$ is the same as the intersection of $\{N(1) = 0\}$ and $\{N(2,5) = 0\}$, which are independent with probabilities $e^{-\lambda}$ and $e^{-3\lambda}$. Hence

$$\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0) = e^{-4\lambda}.$$

b. $N(1,3) = N(1,2) + N(2,3)$, with $N(2,3)$ independent of $N(1,2)$, so $\mathbb{E}(N(1,3) \mid N(1,2) = 3) = 3 + \lambda$.

c. Since $N(2) = 1$, the second interarrival time T_2 has not yet lapsed at $t = 2$. From the memoryless property of the Exponential distribution,

$$\mathbb{E}(T_2 - 2 \mid N(2) = 1) = \frac{1}{\lambda}.$$

Hence the answer is $2 + \lambda^{-1}$.