UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 2

Spring 2024

1. **Two-envelopes puzzle**

   You are handed two envelopes, and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. Professor Jiao claims that the following strategy will increase above $\frac{1}{2}$ your probability of ending up with the envelope with the larger amount: toss a coin repeatedly, let X be equal to $\frac{1}{2}$ plus the number of tosses required to obtain heads for the first time, and switch if the amount in the envelope you selected is less than the value of X. Is he correct?

   Solution: Let $\bar{m}$ and m be the larger and the smaller of the two amounts, respectively. Consider the three events

   $$A = \{X < m\}, \quad B = \{m < X < \bar{m}\}, \quad C = \{\bar{m} < X\}.$$

   Let $\bar{A}$ (or $\bar{B}$ or $\bar{C}$) be the event that A (or B or C, respectively) occurs *and* you first select the envelope containing the larger amount $\bar{m}$. Let A (or B or C) be the event that A (or B or C, respectively) occurs *and* you first select the envelope containing the smaller amount m. Finally, consider the event

   $$W = \{\text{you end up with the envelope containing } \bar{m}\}.$$

   We want to determine $P(W)$ and check whether it is larger than $1/2$ or not. By the total probability theorem, we have

   $$\begin{aligned} P(W|A) &= \frac{1}{2}(P(W|\bar{A}) + P(W|A)) = \frac{1}{2}(1+0) = \frac{1}{2}, \\ P(W|B) &= \frac{1}{2}(P(W|\bar{B}) + P(W|B)) = \frac{1}{2}(1+1) = 1, \\ P(W|C) &= \frac{1}{2}(P(W|\bar{C}) + P(W|C)) = \frac{1}{2}(0+1) = \frac{1}{2}. \end{aligned}$$

   Using these relations together with the total probability theorem, we obtain

   $$\begin{aligned} P(W) &= P(A)P(W|A) + P(B)P(W|B) + P(C)P(W|C) \\ &= \frac{1}{2}(P(A) + P(B) + P(C)) + \frac{1}{2}P(B) \\ &= \frac{1}{2} + \frac{1}{2}P(B). \end{aligned}$$

   Since $P(B) > 0$ by assumption, it follows that $P(W) > 1/2$, so Professor Jiao is correct.

2. **Independence and Pairwise Independence**

A collection of events $\{A_i\}_{i \in I}$ is said to be *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all distinct indices $i \neq j$.

You flip a fair coin 3 times, where the result of each flip is independent of all other flips. For $i = 1, \ldots, 3$, let A_i be the event that the ith flip comes up heads. Let B be the event that in total, an *odd* number of heads are seen. Show that the events $A_1, \ldots, A_3, B$ are pairwise independent but *not* independent.

Solution: A good choice of sample space is $\Omega = \{H, T\}^3$, the set of all 2^3 possible configurations of 3 coin flips, or equivalently $\{0, 1\}^3$, the set of all 3-bit binary strings. We first check that $A_1, \ldots, A_3, B$ are pairwise independent: for $1 \leq i < j \leq 3$,

$$\mathbb{P}(A_i \cap A_j) = \frac{2}{2^3} = \frac{2^2}{2^3} \cdot \frac{2^2}{2^3} = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j).$$

We also see that $\mathbb{P}(B) = \frac{1}{2}$ by symmetry: there is a unique correspondence between outcomes with an odd number of heads and outcomes with an even number of heads, found by simply switching the result of the first flip. $\mathbb{P}(A_i \cap B) = \frac{1}{4}$ by the same argument, now applied to the outcomes in A_i. So, for $1 \leq i \leq 3$,

$$\mathbb{P}(A_i \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A_i) \cdot \mathbb{P}(B).$$

This shows pairwise independence. However, $A_1, \ldots, A_3, B$ are not independent:

$$\mathbb{P}\left(\bigcap_{i=1}^{3} A_i \cap B\right) = \frac{1}{2^3} \neq \left(\frac{1}{2}\right)^4 = \prod_{i=1}^{3} \mathbb{P}(A_i) \cdot \mathbb{P}(B).$$

This formalizes the idea that B is "determined by" the events $A_1, \ldots, A_3$.

3. **Choosing from Any Jar Makes No Difference**

Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w+b)$.

Solution: We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{w+1}{w+b+1}p_i + \frac{w}{w+b+1}(1-p_i) = \frac{1}{w+b+1}p_i + \frac{w}{w+b+1},$$

starting with the initial condition $p_1 = w/(w+b)$. Thus, we have

$$p_2 = \frac{1}{w+b+1} \cdot \frac{w}{w+b} + \frac{w}{w+b+1} = \frac{w}{w+b}.$$

More generally, this calculation shows that if $p_{i-1} = w/(w+b)$, then $p_i = w/(w+b)$. Thus, we obtain $p_i = w/(w+b)$ for all i.

4. **Middle School**

A middle school is composed of 40% sixth graders, 40% seventh graders and 20% eighth graders. The average height of students in these grades are 4, 4.5, and 5 ft. respectively. The variance of heights within each grade are 1, $\frac{1}{2}$, and $\frac{1}{2}$ sq. ft. respectively. Suppose you pick a student at random. Let X denote their grade, and Y denote their height.

a. What is $\mathbb{E}(Y)$?

b. What is $\text{var}(Y)$?

Solution:

a. From the definition of expectation,

$$\mathbb{E}(Y) = \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot \frac{9}{2} + \frac{1}{5} \cdot 5 = \frac{22}{5} = 4.4.$$

b. Y "depends on" X, so it may be hard to find $\text{var}(Y)$ directly. Instead, since we can find $\mathbb{E}(Y \mid X)$ and $\text{var}(Y \mid X)$, we can use the law of total variance:

$$\begin{aligned}
\text{var}(Y) &= \mathbb{E}(\text{var}(Y \mid X)) + \text{var}(\mathbb{E}(Y \mid X)) \\
&= \left(\frac{2}{5} \cdot 1 + \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2}\right) + \left(\frac{2}{5}(4 - 4.4)^2 + \frac{2}{5}(4.5 - 4.4)^2 + \frac{1}{5}(5 - 4.4)^2\right) \\
&= 0.4 + 0.2 + 0.1 + 0.064 + 0.004 + 0.072 \\
&= 0.84.
\end{aligned}$$

5. **Unconscious statistician**

An unconscious statistician comes up with a random variable X with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a and $\mathbb{E}[X]$.

(b) What is the PMF of the random variable $Z = (X - \mathbb{E}[X])^2$?

(c) Using the result from part (b), find the variance of X.

(d) Find the variance of X using the formula $\text{var}(X) = \sum_x (x - \mathbb{E}[X])^2 p_X(x)$.

Solution:

(a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbb{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\text{var}(X) = \mathbb{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7$.

(d) We have

$$\begin{aligned} \text{var}(X) &= \sum_x (x - \mathbb{E}[X])^2 p_X(x) \\ &= 1^2 \cdot (p_X(-1) + p_X(1)) + 2^2 \cdot (p_X(-2) + p_X(2)) + 3^2 \cdot (p_X(-3) + p_X(3)) \\ &= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\ &= 7. \end{aligned}$$

6. **Compact Arrays**

Consider an array of $n \geq 1$ entries, where each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact by moving all the zeros to the end of the array. For example, if we take the array

$$\begin{bmatrix} 6 & 4 & 0 & 0 & 5 & 3 & 0 & 5 & 1 & 3 \end{bmatrix}$$

and make it compact, we now have

$$\begin{bmatrix} 6 & 4 & 5 & 3 & 5 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is nonzero. (The array is indexed starting from 1.) Let X_i be the random variable equal to the index that the ith entry has been moved to after making the array compact. Calculate $\mathbb{E}(X_i)$ and $\operatorname{var}(X_i)$.

Solution: Let Y_j, $j = 1, \ldots, i-1$, be the indicator that the jth entry of the original array is 0. Then the ith entry is moved backwards $\sum_{j=1}^{i-1} Y_j$ positions, so

$$\mathbb{E}(X_i) = i - \sum_{j=1}^{i-1} \mathbb{E}(Y_j) = i - \frac{i-1}{10} = \frac{9i+1}{10}.$$

The variance is also straightforward to compute by the independence of the indicators Y_j. We note that $\operatorname{var}(Y_j) = \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{100}$, so

$$\operatorname{var}(X_i) = \operatorname{var}\left(i - \sum_{j=1}^{i-1} Y_j\right) = \sum_{j=1}^{i-1} \operatorname{var}(Y_j) = \frac{9(i-1)}{100}.$$