

Homework 2

Spring 2024

1. **Two-envelopes puzzle**

You are handed two envelopes, and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. Professor Jiao claims that the following strategy will increase above $\frac{1}{2}$ your probability of ending up with the envelope with the larger amount: toss a coin repeatedly, let X be equal to $\frac{1}{2}$ plus the number of tosses required to obtain heads for the first time, and switch if the amount in the envelope you selected is less than the value of X . Is he correct?

2. Independence and Pairwise Independence

A collection of events $\{A_i\}_{i \in I}$ is said to be *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all distinct indices $i \neq j$.

You flip a fair coin 3 times, where the result of each flip is independent of all other flips. For $i = 1, \dots, 3$, let A_i be the event that the i th flip comes up heads. Let B be the event that in total, an *odd* number of heads are seen. Show that the events A_1, \dots, A_3, B are pairwise independent but *not* independent.

3. Choosing from Any Jar Makes No Difference

Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k . Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w + b)$.

4. Middle School

A middle school is composed of 40% sixth graders, 40% seventh graders and 20% eighth graders. The average height of students in these grades are 4, 4.5, and 5 ft. respectively. The variance of heights within each grade are 1, $\frac{1}{2}$, and $\frac{1}{2}$ sq. ft. respectively. Suppose you pick a student at random. Let X denote their grade, and Y denote their height.

- a. What is $\mathbb{E}(Y)$?
- b. What is $\text{var}(Y)$?

5. Unconscious statistician

An unconscious statistician comes up with a random variable X with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find a and $\mathbb{E}[X]$.
- (b) What is the PMF of the random variable $Z = (X - \mathbb{E}[X])^2$?
- (c) Using the result from part (b), find the variance of X .
- (d) Find the variance of X using the formula $\text{var}(X) = \sum_x (x - \mathbb{E}[X])^2 p_X(x)$.

6. Compact Arrays

Consider an array of $n \geq 1$ entries, where each entry is chosen uniformly randomly from $\{0, \dots, 9\}$. We want to make the array more compact by moving all the zeros to the end of the array. For example, if we take the array

$$[6 \ 4 \ 0 \ 0 \ 5 \ 3 \ 0 \ 5 \ 1 \ 3]$$

and make it compact, we now have

$$[6 \ 4 \ 5 \ 3 \ 5 \ 1 \ 3 \ 0 \ 0 \ 0]$$

Let i be a fixed positive integer in $\{1, \dots, n\}$. Suppose that the i th entry of the array is nonzero. (The array is indexed starting from 1.) Let X_i be the random variable equal to the index that the i th entry has been moved to after making the array compact. Calculate $\mathbb{E}(X_i)$ and $\text{var}(X_i)$.