UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 3
Spring 2024

1. **Stock trader**

   A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period, and assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying

   $$-2 \le x \le 4, \quad -1 \le y - x \le 1.$$

   (a) Find the marginal PMFs and the means of X and Y.
   (b) Find the mean of the trader's profit.

   Solution:

   (a) There are 21 integer pairs (x, y) in the region

   $$R = \{(x, y) \mid -2 \le x \le 4, -1 \le y - x \le 1\},$$

   so that the joint PMF of X and Y is

   $$p_{X,Y}(x, y) = \begin{cases} \frac{1}{21}, & \text{if } (x, y) \text{ is in } R, \\ 0, & \text{otherwise.} \end{cases}$$

   For each x in the range $[-2, 4]$, there are three possible values of Y. Thus, we have

   $$p_X(x) = \begin{cases} \frac{3}{21}, & \text{if } x = -2, -1, 0, 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

   The mean of X is the midpoint of the range $[-2, 4]$:

   $$\mathbb{E}[X] = 1.$$

   The marginal PMF of Y is obtained by using the tabular method. We have

   $$p_Y(y) = \begin{cases} \frac{1}{21}, & \text{if } y = -3, \\ \frac{2}{21}, & \text{if } y = -2, \\ \frac{3}{21}, & \text{if } y = -1, 0, 1, 2, 3, \\ \frac{2}{21}, & \text{if } y = 4, \\ \frac{1}{21}, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

   The mean of Y is

   $$\mathbb{E}[Y] = \frac{1}{21} \cdot (-3 + 5) + \frac{2}{21} \cdot (-2 + 4) + \frac{3}{21} \cdot (-1 + 1 + 2 + 3) = 1.$$

   (b) The profit is given by

   $$P = 100X + 200Y,$$

   so that

   $$\mathbb{E}[P] = 100 \cdot \mathbb{E}[X] + 200 \cdot \mathbb{E}[Y] = 100 \cdot 1 + 200 \cdot 1 = 300.$$

2. **Poisson Practice**

Suppose X is a Poisson random variable with parameter λ. Find the following:

a. $\mathbb{E}(X^2)$.

b. $\mathbb{P}(X \text{ is even})$. (*Hint*: Use the Taylor series expansion of e^x.)

Solution:

a. First compute

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_{x=2}^{\infty} x(x-1)\frac{\lambda^x \varepsilon^{-\lambda}}{x!} = \sum_{x=2}^{\infty} \frac{\lambda^x \varepsilon^{-\lambda}}{(x-2)!} = \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}\varepsilon^{-\lambda}}{(x-2)!} \\ &= \lambda^2.\end{aligned}$$

Hence, $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \lambda^2 + \lambda$.

b. Note that

$$\begin{aligned}\Pr(X \text{ is even}) &= \sum_{k=0}^{\infty} \Pr(X = 2k) = \sum_{k=0}^{\infty} \frac{\lambda^{2k}\varepsilon^{-\lambda}}{(2k)!} \\ &= \frac{\varepsilon^{-\lambda}}{2}\Big(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}\Big) = \frac{\varepsilon^{-\lambda}}{2}(\varepsilon^{\lambda} + \varepsilon^{-\lambda}) \\ &= \frac{1 + \varepsilon^{-2\lambda}}{2}.\end{aligned}$$

To explain the second line, note that the odd terms cancel out and the even terms are counted twice.

3. **Poisson Properties**

a. Suppose X and Y are independent Poisson random variables with means λ and μ respectively. Prove that $X+Y$ has the Poisson distribution with mean $\lambda+\mu$. **Note**: It is *not* enough to use linearity of expectation to say that $X+Y$ has mean $\lambda+\mu$. You are asked to prove that the *distribution* of $X+Y$ is Poisson.

b. Given X and Y as above, what is the distribution of X conditioned on $X+Y=z$, $z\in\mathbb{N}$?

Solution:

a. The distribution of the sum of two independent random variables is the *convolution* of their individual distributions. For $z\in\mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}(X+Y=z) &= \sum_{x=0}^{z}\mathbb{P}(X=x)\cdot\mathbb{P}(Y=z-x)\\ &= \sum_{x=0}^{z}\frac{\lambda^x}{x!}e^{-\lambda}\cdot\frac{\mu^{z-x}}{(z-x)!}e^{-\mu}\\ &= \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z}\frac{z!}{x!(z-x)!}\lambda^x\mu^{z-x}\\ &= \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z}\binom{z}{x}\lambda^x\mu^{z-x}\\ &= \frac{e^{-(\lambda+\mu)}}{z!}(\lambda+\mu)^z,\end{aligned}$$

which shows that $X+Y\sim\text{Poisson}(\lambda+\mu)$.

b.

$$\begin{aligned}\mathbb{P}(X=x\mid X+Y=z) &= \frac{\mathbb{P}(X=x,X+Y=z)}{\mathbb{P}(X+Y=z)}\\ &= \mathbb{P}(X=x)\cdot\mathbb{P}(Y=z-x)\Big/\left(\frac{(\lambda+\mu)^z}{z!}e^{-(\lambda+\mu)}\right)\\ &= \frac{\lambda^x}{x!}e^{-\lambda}\cdot\frac{\mu^{z-x}}{(z-x)!}e^{-\mu}\Big/\left(\frac{(\lambda+\mu)^z}{z!}e^{-(\lambda+\mu)}\right)\\ &= \binom{z}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^{z-x},\end{aligned}$$

which shows that $X\mid X+Y=z\sim\text{Binomial}(z,\frac{\lambda}{\lambda+\mu})$.

Remark: These properties will be used extensively when we discuss the Poisson process.

4. **PDF Practice**

Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{x}{4}, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 2\}$.

(a) Find $\mathbb{E}[X]$, $P(A)$, $f_{X|A}(x)$, and $\mathbb{E}[X|A]$.

(b) Let $Y = X^2$. Find $\mathbb{E}[Y]$ and $\text{var}(Y)$.

Solution:

(a) We have

$$\mathbb{E}[X] = \int_1^3 \frac{x^2}{4}\,dx = \frac{x^3}{12}\Big|_1^3 = \frac{27}{12} - \frac{1}{12} = \frac{26}{12} = \frac{13}{6},$$

and

$$P(A) = \int_2^3 \frac{x}{4}\,dx = \frac{x^2}{8}\Big|_2^3 = \frac{9}{8} - \frac{4}{8} = \frac{5}{8}.$$

We also have

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)}, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

which simplifies to

$$f_{X|A}(x) = \begin{cases} \frac{2x}{5}, & \text{if } 2 \leq x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

from which we obtain

$$\mathbb{E}[X|A] = \int_2^3 x \cdot \frac{2x}{5}\,dx = \frac{2x^3}{15}\Big|_2^3 = \frac{54}{15} - \frac{16}{15} = \frac{38}{15}.$$

(b) We have

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_1^3 \frac{x^3}{4}\,dx = 5,$$

and

$$\mathbb{E}[Y^2] = \mathbb{E}[X^4] = \int_1^3 \frac{x^5}{4}\,dx = \frac{91}{3}.$$

Thus,

$$\text{var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{91}{3} - 5^2 = \frac{16}{3}.$$

5. **Joint Density for Exponential Distribution**

a. If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are independent, compute $\mathbb{P}(X < Y)$.

b. If $X_1, \dots, X_n$ are independent and Exponentially distributed with parameters $\lambda_1, \dots, \lambda_n$, show that $\min_{1 \le k \le n} X_k \sim \text{Exponential}(\sum_{j=1}^n \lambda_j)$.

c. Deduce that

$$\mathbb{P}\left(X_i = \min_{1 \le k \le n} X_k\right) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}.$$

Solution:

a. By the law of total probability,

$$\mathbb{P}(X < Y) = \int_0^\infty \mathbb{P}(X < y \mid Y = y) \cdot f_Y(y) \, \mathrm{d}y.$$

Since X and Y are independent, $\mathbb{P}(X < y \mid Y = y) = \mathbb{P}(X < y)$. Plugging in the known $\mathbb{P}(X < y) = 1 - e^{-\lambda y}$ and $f_Y(y) = \mu e^{-\mu y}$, we get

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}.$$

b. The ccdf of $X := \min_{1 \le k \le n} X_k$ is precisely the ccdf of an $\text{Exponential}(\sum_{j=1}^n \lambda_j)$:

$$\mathbb{P}(X \ge x) = \mathbb{P}(X_1 \ge x, \dots, X_n \ge x) = \prod_{k=1}^n \mathbb{P}(X_k \ge x) = \prod_{k=1}^n e^{-\lambda_k x} = e^{-x \sum_{k=1}^n \lambda_k}.$$

c. Now, we observe that

$$\mathbb{P}\left(X_i = \min_{1 \le k \le n} X_k\right) = \mathbb{P}\left(X_i \le \min_{k \ne i} X_k\right).$$

By part b, $\min_{k \ne i} X_k \sim \text{Exponential}(\sum_{j \ne i} \lambda_j)$. Then, by part a, the claim follows.

6. **Waiting time**

Calamity Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time? Is this random variable discrete or continuous?

Solution: Let X be the waiting time and Y be the number of customers found. For $x < 0$, we have $F_X(x) = 0$, while for $x \geq 0$,

$$F_X(x) = \frac{1}{2}P(X \leq x|Y = 0) + \frac{1}{2}P(X \leq x|Y = 1).$$

Since

$$P(X \leq x|Y = 0) = 1,$$

and

$$P(X \leq x|Y = 1) = 1 - e^{-\lambda x},$$

we obtain

$$F_X(x) = \begin{cases} \frac{1}{2}(2 - e^{-\lambda x}), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the CDF has a discontinuity at $x = 0$. The random variable X is neither discrete nor continuous.