UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 4
Spring 2024

1. **Change of Variables**

   Let X be a continuous random variable with cdf F_X and pdf $f_X > 0$ everywhere, and let $Y = g(X)$, where g is a differentiable function.

   a. Suppose that g is also invertible. Find the pdf of Y, f_Y, in terms of g and f_X.

   b. Let $U \sim \text{Uniform}([0,1])$. Using the conclusion from part a, show that $F_X^{-1}(U)$ has the same distribution as X. (This allows us to generate a given random variable given only a uniform random number generator.)

   c. Now suppose that $g(x) = x^2$. Find the pdf of Y in terms of the pdf of X. Also find the pdf of Y when X is a standard normal random variable in particular.

      (Note that this g is not invertible, unlike in part a.)

2. **Convolution practice**

   The random variables X, Y, and Z are independent and uniformly distributed between zero and one. Find the PDF of X + Y + Z.

3. **Moment-Generating Functions Practice**

The **moment-generating function** (mgf) of a random variable X is the function

$$M_X(s) = \mathbb{E}(e^{sX}) = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{(sX)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\,\mathbb{E}(X^k).$$

In this class, we will not worry about technical details about the convergence of Taylor series, so we will say that the mgf is equal to any of the expressions above.

The mgf gets its name because it is the function *generating* the moments $\mathbb{E}(X^p)$, $p \geq 1$, of X. More specifically, by evaluating the pth derivative of the mgf at $s = 0$, we have a method to explicitly find the pth moment of X from its mgf:

$$\left[\frac{\mathrm{d}^p}{\mathrm{d}s^p} M_X(s)\right]_{s=0} = \left[\sum_{k=p}^{\infty} \frac{s^{k-p}}{p!}\,\mathbb{E}(X^k)\right]_{s=0} = \mathbb{E}(X^p) + \sum_{k=p+1}^{\infty} 0 = \mathbb{E}(X^p).$$

Consider a random variable Z with moment-generating function

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8} \qquad \text{for } |s| < 2.$$

Calculate the following quantities:

a. The numerical value of the parameter a.

b. $\mathbb{E}(Z)$.

c. $\mathrm{var}(Z)$.

4. **Revisiting Proofs Using Transforms**

   a. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ be independent. Calculate the MGF of $X + Y$, and use this to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$.

   b. Calculate the MGF of $X \sim \text{Exponential}(\lambda)$, and use this to find all of the moments of X.

   c. Repeat the above part, but for $X \sim \mathcal{N}(0, 1)$.

5. **Kelly strategy**

Consider a gambler who at each gamble either wins or loses his bet with probabilities p and $1-p$, independent of earlier gambles. When $p > 1/2$, a popular gambling system, known as the Kelly strategy, is to always bet the fraction $2p-1$ of the current fortune. Compute the expected fortune after n gambles, starting with x units and employing the Kelly strategy

6. **Covariance Matrix**

For a random vector $X = [X_1, X_2, \ldots, X_n]^\mathsf{T}$, its covariance matrix Σ is defined with entries $\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$. Suppose that $\mathbb{E}[X] = 0$.

a. Show that Σ is positive semi-definite, i.e. for all $v \in \mathbb{R}^n$, we have $v^\mathsf{T} \Sigma v \geq 0$.
b. Show that if the X_i's are *pairwise* independent, then Σ is diagonal.
c. Give an example of two random variables X_1, X_2 with a diagonal covariance matrix, but such that X_1, X_2 are not independent.