UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 6
Spring 2024

1. **Convergence in Probability**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1,1]$. Show that the following sequences $(Y_n)_{n\in\mathbb{N}}$ converge in probability to some limit.

a. $Y_n = \prod_{i=1}^n X_i$.
b. $Y_n = \max\{X_1,\dots,X_n\}$.
c. $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

Solution:

a. By the independence of the random variables,

$$\mathbb{E}(Y_n) = \mathbb{E}(X_1)\cdots\mathbb{E}(X_n) = 0$$
$$\mathrm{var}(Y_n) = \mathbb{E}(Y_n^2) = (\mathrm{var}(X_1))^n = \left(\frac{1}{3}\right)^n.$$

Since $\mathrm{var}(Y_n) \to 0$ as $n \to \infty$, by Chebyshev's inequality, the sequence converges to its mean 0 in probability.

b. Consider $\varepsilon \in (0,1]$. We see that

$$\begin{aligned}\mathbb{P}(|Y_n - 1| \ge \varepsilon) &= \mathbb{P}(\max\{X_1,\dots,X_n\} \le 1-\varepsilon)\\ &= \mathbb{P}(X_1 \le 1-\varepsilon,\dots,X_n \le 1-\varepsilon)\\ &= \mathbb{P}(X_1 \le 1-\varepsilon)^n\\ &= \left(1-\frac{\varepsilon}{2}\right)^n,\end{aligned}$$

so $\mathbb{P}(|Y_n - 1| \ge \varepsilon) \to 0$ as $n \to \infty$, and we are done.

c. We can find the expectation, then bound the variance:

$$\mathbb{E}(Y_n) = \frac{1}{n}\cdot n\,\mathbb{E}(X_1^2) = \frac{1}{3},$$
$$\mathrm{var}(Y_n) = \frac{1}{n}\,\mathrm{var}(X_1^2) \le \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,$$

since $X_1^2 \le 1$. Hence, we see that $Y_n \to \frac{1}{3}$ in probability as $n \to \infty$.

2. **Bernoulli Convergence**

Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

a. Show that X_n converges to 0 in probability.

b. Argue that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

c. Using part b, show that X_n does **not** converge almost surely to 0.
Hint: Consider applying the union bound and the independence of the X_n.

Solution:

a. We want to show that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) = 0.$$

Because each X_n can only be 0 or 1, if $\varepsilon \geq 1$, then $\mathbb{P}(|X_n - 0| > \varepsilon) = 0$, so the limit is also zero. If $0 < \varepsilon < 1$, then

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n = 1) = \frac{1}{n} \to 0.$$

b. Since each X_n can only take on the values 0 or 1, the limit of X_n is 0 iff the sequence $X_1, X_2, \ldots$ is eventually always 0. In other words, $\{\lim_{n\to\infty} X_n = 0\}$ occurs if and only if there exists an N such that for all $n \geq N$, $X_n = 0$. Thus

$$\left\{\lim_{n\to\infty} X_n = 0\right\} = \bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}.$$

c. Applying the union bound to the equality in part b,

$$\begin{aligned}
\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) &\leq \sum_{N=1}^{\infty} \mathbb{P}(X_n = 0 \text{ for all } n \geq N) \\
&= \sum_{N=1}^{\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} \{X_n = 0\}\right)
\end{aligned}$$

Because the X_n are independent, this equals

$$\begin{aligned}
&= \sum_{N=1}^{\infty} \prod_{n=N}^{\infty} \mathbb{P}(X_n = 0) \\
&= \sum_{N=1}^{\infty} \frac{N-1}{N} \cdot \frac{N}{N+1} \cdot \frac{N+1}{N+2} \cdots
\end{aligned}$$

By telescoping, this infinite product is zero for any value of N, so we have

$$= \sum_{N=1}^{\infty} 0 = 0.$$

Since this probability is not 1, X_n does not converge almost surely to 0. In fact, since this probability is 0, X_n *almost surely does not converge* to 0. A related result is Kolmogorov's 0–1 law, which states that a sequence of independent random variables either converges or does not converge with probability 1.

3. **More Almost Sure Convergence**

   a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

   b. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

   Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

Solution:

   a. Yes. If a sequence oscillates between two distinct values infinitely often, then it does not converge. Here, we have a sequence that oscillates infinitely often, with probability 1, which means that the sequence in fact **diverges** with probability 1.

   The above may have been very cumbersome to read, which is why we often abbreviate "with probability 1" with "a.s." Then the above reads "$(X_n)_{n\in\mathbb{N}}$ oscillates between two values infinitely often a.s., so $(X_n)_{n\in\mathbb{N}}$ does not converge a.s."

   b. Yes. Observe that when $Y = y \neq 0$, $(X_n)_{n\in\mathbb{N}}$ will converge to y^{-1}, but when $Y = 0$, $(X_n)_{n\in\mathbb{N}}$ does not converge. However, $\mathbb{P}(Y = 0) = 0$, since Y is a continuous random variable. In other words,

$$\mathbb{P}(X_n \text{ does not converge as } n \to \infty) = \mathbb{P}(Y = 0) = 0$$
$$\mathbb{P}(X_n \text{ converges as } n \to \infty) = \mathbb{P}(Y \neq 0) = 1,$$

   so $(X_n)_{n\in\mathbb{N}}$ converges a.s.

   c. No. The sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between 0 and successively higher powers of two infinitely often a.s., so it does not converge a.s.

   d. Yes. Fix $\varepsilon > 0$. For $n \in \mathbb{Z}^+$, one has

$$\mathbb{P}(|X_n| > \varepsilon) = \frac{1}{2^k},$$

   where $k = \lfloor \log_2 n \rfloor$. As $n \to \infty$, the above probability goes to 0, so $X_n \to 0$ in probability. Intuitively, $(X_n)_{n\in\mathbb{N}}$ has infinitely many oscillations, so it cannot converge a.s. However, the probability of each oscillation shrinks to 0, so $(X_n)_{n\in\mathbb{N}}$ converges in probability.

   The expectations do not converge. For all n, one has $\mathbb{E}(X_n) = 1$, so it is not the case that $\mathbb{E}(X_n) \to 0$ as $n \to \infty$. Hence, convergence in probability is not sufficient to imply that the expectations converge. In fact, almost sure convergence is not sufficient either.

4. **Convergence in L^p**

Let $p \geq 1$. A sequence of random variables $(X_n)_{n\geq 1}$ is said to **converge in** L^p (norm) to a random variable X if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0.$$

Prove that if $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Solution: Note that for $p \geq 1$, $x \mapsto x^p$ is a monotonic function. By Markov's inequality,

$$\mathbb{P}(|X_n - X| \geq \varepsilon) = \mathbb{P}(|X_n - X|^p \geq \varepsilon^p) \leq \frac{\mathbb{E}(|X_n - X|^p)}{\varepsilon^p}.$$

If $X_n \to X$ in L^p, i.e. $\mathbb{E}(|X_n - X|^p) \to 0$, then $\mathbb{P}(|X_n - X| \geq \varepsilon) \to 0$ for any $\varepsilon > 0$, which is precisely convergence in probability.

5. **Sum of Rolls**

You roll a fair 6-sided die 100 times, and you call the sum of the values of all your rolls X. Use the Central Limit Theorem to approximate the probability that $X > 400$. You may use a calculator and Gaussian lookup table.

Solution: The value of an individual roll, distributed as Uniform([6]), has mean 3.5 and variance $\frac{6^2-1}{12} = \frac{35}{12}$. Call $\sigma = \sqrt{\frac{35}{12}}$. Then, by the Central Limit Theorem, $(X - 350)/(10\sigma)$ is approximately $\mathcal{N}(0,1)$ distributed, i.e. X is approximately $\mathcal{N}(350, 100\sigma^2)$. Therefore

$$\begin{aligned}\mathbb{P}(X > 400) &\approx \mathbb{P}(\mathcal{N}(350, 100\sigma^2) > 400)\\ &= \mathbb{P}(\mathcal{N}(0,1) > \tfrac{400-350}{10\sigma})\\ &\approx 1 - \Phi(2.93)\\ &= 1 - 0.9983 = 0.0017.\end{aligned}$$

6. **CLT Cannot Be Upgraded**

a. Show that if X_n converges to X in probability and Y_n to Y in probability, then $aX_n + Y_n$ converges to $aX + Y$ in probability.

b. Show that the CLT cannot be upgraded to convergence in probability or almost surely. That is, if $X_1, X_2, \ldots$ are i.i.d. with mean 0 and variance 1, prove that it cannot be the case that

$$Z_n \coloneqq \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to Z, \quad \text{where } Z \sim \mathcal{N}(0,1), \text{ almost surely or in probability.}$$

Hint: From part a, the sequence of random variables $\sqrt{2}Z_{2n} - Z_n$ converges in probability to $(\sqrt{2}-1)Z$. Does this contradict the fact that Z_n converges to Z in probability?

Solution:

a. Let $\varepsilon > 0$. By the union bound,

$$\begin{aligned}\mathbb{P}(|(aX+Y) - (aX_n + Y_n)| > \varepsilon) &\le \mathbb{P}(|a(X - X_n)| > \varepsilon/2 \text{ or } |Y - Y_n| > \varepsilon/2) \\ &\le \mathbb{P}(|X - X_n| > \varepsilon/(2|a|)) + \mathbb{P}(|Y - Y_n| > \varepsilon/2),\end{aligned}$$

which we know converges to 0. Hence $aX_n + Y_n$ converges to $aX + Y$ in probability.

b. We observe that

$$\sqrt{2}Z_{2n} - Z_n = \frac{X_{n+1} + X_{n+2} + \cdots + X_{2n}}{\sqrt{n}},$$

is equal in distribution to Z_n, and hence must converge in distribution to Z. However, convergence in probability implies convergence in distribution, so $\sqrt{2}Z_{2n} - Z_n$ must also converge to $(\sqrt{2}-1)Z$ in distribution. As $Z \neq (\sqrt{2}-1)Z$, this is a contradiction.

Lastly, as Z_n cannot converge in probability, it cannot converge almost surely either, since almost sure convergence is a stronger form of convergence.