UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 6

Spring 2024

1. **Convergence in Probability**

   Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences $(Y_n)_{n\in\mathbb{N}}$ converge in probability to some limit.

   a. $Y_n = \prod_{i=1}^{n} X_i$.

   b. $Y_n = \max\{X_1, \ldots, X_n\}$.

   c. $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

2. **Bernoulli Convergence**

   Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

   a. Show that X_n converges to 0 in probability.

   b. Argue that

   $$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

   c. Using part b, show that X_n does **not** converge almost surely to 0.
   Hint: Consider applying the union bound and the independence of the X_n.

3. **More Almost Sure Convergence**

a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

b. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \ldots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

4. **Convergence in L^p**

Let $p \geq 1$. A sequence of random variables $(X_n)_{n\geq 1}$ is said to **converge in** L^p (norm) to a random variable X if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0.$$

Prove that if $X_n \to X$ in L^p, then $X_n \to X$ in probability.

5. **Sum of Rolls**

You roll a fair 6-sided die 100 times, and you call the sum of the values of all your rolls X. Use the Central Limit Theorem to approximate the probability that $X > 400$. You may use a calculator and Gaussian lookup table.

6. **CLT Cannot Be Upgraded**

a. Show that if X_n converges to X in probability and Y_n to Y in probability, then $aX_n + Y_n$ converges to $aX + Y$ in probability.

b. Show that the CLT cannot be upgraded to convergence in probability or almost surely. That is, if $X_1, X_2, \ldots$ are i.i.d. with mean 0 and variance 1, prove that it cannot be the case that

$$Z_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to Z, \quad \text{where } Z \sim \mathcal{N}(0,1), \text{ almost surely or in probability.}$$

Hint: From part a, the sequence of random variables $\sqrt{2}Z_{2n} - Z_n$ converges in probability to $(\sqrt{2}-1)Z$. Does this contradict the fact that Z_n converges to Z in probability?