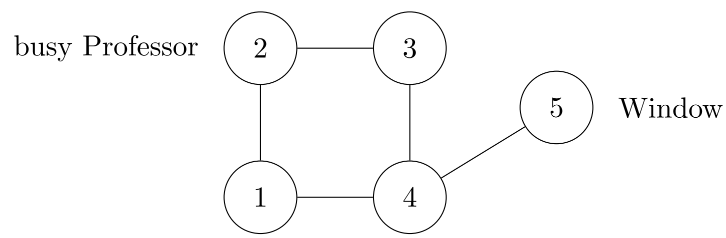
UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 7
Spring 2024

1. **Fly on a Graph**

   A fly wanders around on a graph G with vertices $V = \{1, \ldots, 5\}$, as shown below.



   a. Suppose that the fly wanders as follows: if it is at node i at time n, then it chooses one of its neighbors j of i uniformly at random, and then wanders to node j at time $n+1$. For times $n = 0, 1, 2, \ldots$, let X_n be the fly's position at time n. Argue that $(X_n)_{n \in \mathbb{N}}$ is a Markov chain, and find the invariant distribution.

   b. For the process in part a, suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very busy, so they don't move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

   c. Now suppose that the fly wanders as follows: when it is at node i at time n, it chooses uniformly from all neighbors of node i except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let Y_n be the fly's position at time n. Following the story in the previous part with the professor and window, once the fly reaches node 2 and 5, it remains at that state.

      Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains and define new states so that it is a valid Markov chain.

2. **Higher-Order Markov Chains**

   Let k be a fixed positive integer. A stochastic process $(X_n)_{n\in\mathbb{N}}$ taking values in a discrete state space $\mathcal{X}$ is called a k**th order (time homogeneous) Markov chain** if for all $n \in \mathbb{N}$ and all feasible sequences $x_0, x_1, \ldots, x_{n+k} \in \mathcal{X}$,

$$
\begin{aligned}
&\Pr(X_{n+k} = x_{n+k} \mid X_0 = x_0, X_1 = x_1, \ldots, X_{n+k-1} = x_{n+k-1}) \\
&\quad = \Pr(X_{n+k} = x_{n+k} \mid X_n = x_n, \ldots, X_{n+k-1} = x_{n+k-1}) \\
&\quad = P_k(x_{n+k} \mid x_n, \ldots, x_{n+k-1}).
\end{aligned}
$$

   In other words, the transition to the next state depends only on the previous k states. For example, if X_n represents the position of a particle moving with constant velocity at time n, then the system is a second-order Markov chain because the previous two position measurements are needed to infer the particle's velocity.

   Show that we can "embed" $(X_n)_{n\in\mathbb{N}}$ into a *first-order* Markov chain $(Z_n)_{n\in\mathbb{N}}$ with an augmented state space, in the sense that X_n can be recovered from Z_n. This allows us to apply algorithms such as the Viterbi algorithm to systems with higher orders of dependence.

3. **Doubly Stochastic Matrix**

   A matrix is called **doubly stochastic** if all of its entries are nonnegative, and each row and each column sums to 1. Show that any doubly stochastic matrix is a valid transition probability matrix for a Markov chain. Then, prove that the stationary distribution for a doubly stochastic irreducible matrix is uniform over the state space.

4. **Moving Books Around**

You have N books labelled $1, \ldots, N$ on your shelf. At each time step, you pick a book i with probability $\frac{1}{N}$, place it on the left of all others on the shelf, then repeat this process, each step independent of any other step. Construct a suitable Markov chain which takes values in the set of all $N!$ permutations of the books.

a. Find the transition probabilities of the Markov chain.

b. Find its stationary distribution.

   Hint: You can guess the stationary distribution before computing it.

5. **Terminating Markov Chain**

Consider a Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}.$$

a. Compute $P^n_{i,i}$ for all $i = 0, 1, 2$ and $n \geq 1$.

b. Compute $P^n_{0,1}$ and $P^n_{1,2}$ for $n \geq 1$.

c. What is the probability that the chain lands in state 2 in at most four steps starting from state 0?

d. Given that $X_0 = 0$, what is the distribution of X_n?

e. Identify an eigenvector for P^T. What is its associated eigenvalue?

f. Let T be the number of time steps, starting from state 0, until the chain reaches state 2. Find $\mathbb{E}(T)$. What is the distribution of T?

6. **Hitting Time with Coins**

   Consider a sequence of fair coin flips.

   a. What is the expected number of flips until we first see two heads in a row?
   b. What is the expected number of flips until we see a head followed immediately by a tail?