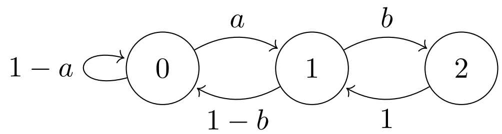
37. **Three-State Chain**

Consider the following Markov chain, where $0 < a, b < 1$.



a. Calculate $\mathbb{P}(X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1 \mid X_0 = 0)$.

b. Show that the Markov chain is irreducible and aperiodic.

c. Find the invariant or stationary distribution.

Solution:

a. By the Markov property, this probability is

$$P(0,1) \cdot P(1,0) \cdot P(0,0) \cdot P(0,1) = a \cdot (1-b) \cdot (1-a) \cdot a = a^2(1-a)(1-b).$$

b. The chain is irreducible because its transition diagram is strongly connected — there is a path from any state to any other state — and it is aperiodic because there is a self-loop.

c. To find the stationary distribution, let us solve the balance equations:

$$\pi(2) = b\pi(1),\ \pi(1) = a\pi(0) + \pi(2),\ \pi(0) + \pi(1) + \pi(2) = 1,$$

from which we find the solution

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{1-b+a+ab}\begin{bmatrix}1-b & a & ab\end{bmatrix}.$$

4 Basics of Markov Chain [21 points]

Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$\begin{pmatrix} 1/2 & 0 & 1/8 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/9 & 5/9 & 2/9 \\ 0 & 0 & 0 & 0 & 5/6 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 2/3 & 1/3 & 0 \end{pmatrix}$$

(a) Write down your SID on the top right corner to get 3 points. (3 points)

(b) Find $\mathbb{P}(X_6 = 4 | X_3 = 2, X_2 = 0)$. (3 points)

(c) Find the recurrent and transient classes. (6 points)

(d) Consider the Markov chain restricted on the recurrent class, is it reversible? Prove your claim. (6 points)

(e) Suppose the initial state is 0, find the expected total number of visits to state 3, i.e.,

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}(X_n = 3) | X_0 = 0\right]$$

(3 points)

(a) SID (3 points).

(b) We have

$$\mathbb{P}(X_6 = 4 | X_3 = 2, X_2 = 0) = \mathbb{P}(X_6 = 4 | X_5 = 0) = 1/8. \text{ (3 points)}$$

(c) Transient class: (0,1,2,3). (3 points)

Recurrent class: (4,5,6). (3 points)

(d) Yes (2 points). The invariant distribution is $(1/2, 1/3, 1/6)$(2 points), which verifies the detailed balance condition. (2 points)

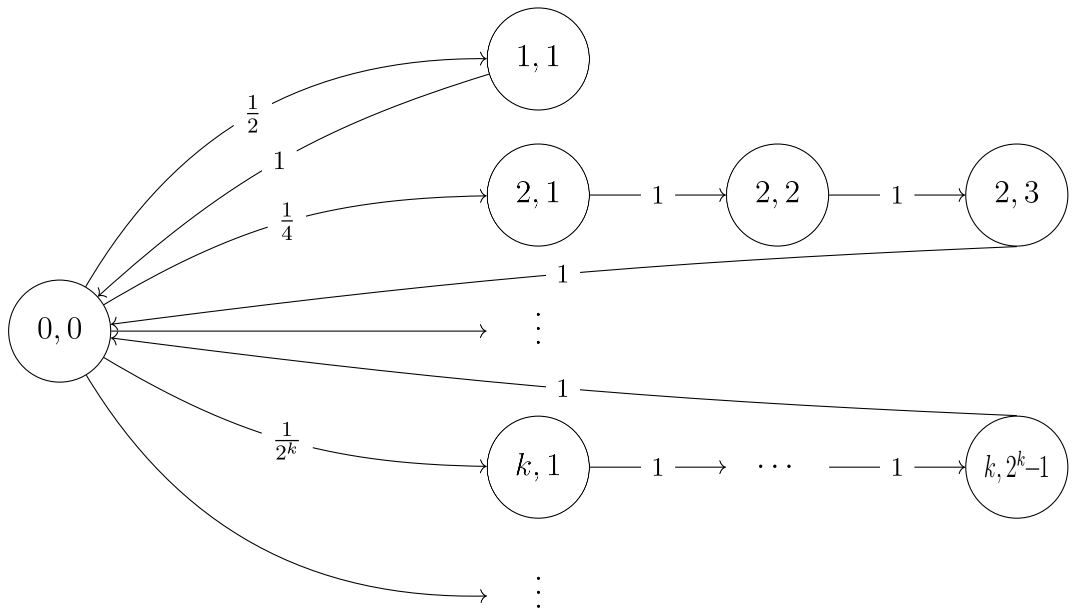
(e) By first-step equation, we have $\mathbb{P}(\tau_3 < \infty | X_0 = 0) = 3/4$. It follows that

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}(X_n = 3) | X_0 = 0\right] = 3. \text{ (3 points)}$$

.

6 Bot on a Stroll [30 points]

The EECS 126 Bot is taking a walk on a Markov chain with state space $\mathbb{N} \times \mathbb{N}$, starting from state $(0, 0)$, as shown by the graph below.



From state $(0, 0)$, the bot chooses "path k" with probability 2^{-k} for $k = 1, 2, \ldots$ Each path k contains $2^k - 1$ states, which the bot will travel through in sequence then return to $(0, 0)$ deterministically.

(a) Is this Markov chain irreducible? Justify your answer.

(b) What is the period of this Markov chain?

(c) What is the expected time to return to state $(0, 0)$?

(d) Is this Markov chain positive recurrent, null recurrent, or transient? Justify your answer.

(a) The Markov chain is irreducible. The path from state (x_t, y_t) to (x_{t+1}, y_{t+1}) can be traced out by first following path x_t to reach $(0, 0)$ in $2^{x_t} - y_t$ steps, then jumping to $(x_{t+1}, 1)$ and following path x_{t+1} for $y_{t+1} - x_{t+1}$ steps.

(b) The period is 2 since the the time to travel path k is 2^k for $k = 1, 2, 3, \ldots$ with a GCD of 2.

(c) The probability to take path i is 2^{-i}, and the time it takes to return to $(0, 0)$ given that we take path i is $(2^i - 1) + 1 = 2^i$. Thus, the expected return time is

$$\sum_{i=1}^{\infty} 2^i \cdot 2^{-i} = \sum_{i=1}^{\infty} 1 = \infty.$$

(d) The Markov chain is null recurrent. From $(0,0)$, no matter which path i the bot takes, it will always return to $(0,0)$ in $2^i < \infty$ steps. Since it always returns to state $(0,0)$, the Markov chain is recurrent. From the previous part, since the expected return time is infinite, we further classify the Markov chain as null recurrent.

4 Double Heads [10 + 10 points]

Suppose you flip a fair coin repeatedly.

a) Given that the first flip turns heads, what is the expected number of tosses (including your first toss) you need to make until you see two heads in a row?

b) Again, given that the first flip turns heads, what is the probability you see two heads in a row before you see two tails in a row?

a) We can model this problem as a DTMC, with state space $\{HH, HT, TH, TT\}$ and transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

where the columns (rows) are respectively correspond to states HH, HT, TH, TT. The state $s_1 s_2$ means we saw s_2 on the most recent flip, and s_1 on the flip before that. Now, we can imagine our first flip turning heads starts us in state HH, and we should compute the average number of flips to return to HH; the number of flips we are after is then 1 plus this number. By inspection, P admits stationary distribution $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ (you can also observe P is doubly stochastic). This chain is irreducible, so by the big theorem, the average number of flips to see two heads in a row, given our first flip is heads, is $1 + 4 = 5$.

b) We can do this by appealing to the equations for hitting probabilities. Set $h(HH) = 1$, and $h(TT) = 0$, where $h(s)$ equals the probability we'll hit HH before TT, given we start in state s for the DTMC formulated in part (a). The hitting probability equations for states $h(HT)$ and $h(TH)$ are

$$h(HT) = \frac{1}{2}h(TH) + \frac{1}{2}h(TT) = \frac{1}{2}h(TH)$$
$$h(TH) = \frac{1}{2}h(HT) + \frac{1}{2}h(HH) = \frac{1}{2}h(HT) + \frac{1}{2} = \frac{1}{4}h(TH) + \frac{1}{2}.$$

The last says $h(TH) = \frac{2}{3}$, which gives $h(HT) = \frac{1}{3}$. Since we flipped a heads on the first toss, we can imagine starting in state HT or HH, with probability $\frac{1}{2}$ each, determined by our second flip. The probability we'll see HH before TT is thus equal to

$$P\{\text{we see } HH \text{ before } TT \mid \text{first flip is } H\} = \frac{1}{2}h(HT) + \frac{1}{2}h(HH) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$$

5 Ehrenfest's diffusion model [21 points]

In Ehrenfest's diffusion model, a container is separated by a permeable membrane in the middle and filled with a total of K particles. At each time $n = 1, 2, \dots$, one particle is picked uniformly in random among the K particles and placed into the other part of the container. Let X_n be the number of particles in the left part of the container.

(a) Write down your SID on the top right corner to get 3 points. (3 points)

(b) What is the transition probability $\mathbf{P}$ of Ehrenfest's diffusion model? (6 points)

(c) Find the stationary distribution. (6 points)

(d) Suppose initially all K particles are located at the right part of the container. For each of the following sequences of random variables, study its convergence. (6 points)

That is, you are expected to answer: (i) if the sequence converges; (ii) if so, what is the limit, and what is the notion of convergence (e.g. almost surely, in probability, or in distribution); (iii) if not, briefly explain the reason.

- $(X_n)_{n=1,2,\dots}$
- $(Y_n)_{n=1,2,\dots}$ where $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$

(a) SID (3 points).

(b) The transition probability is given by

$$p_{ij} = \begin{cases} \frac{i}{K}, & j = i-1 \text{ (2 points)} \\ \frac{K-i}{K}, & j = i+1 \text{ (2 points)} \\ 0, & \text{else (2 points)} \end{cases}$$

(c) We have the following relation:

$$\pi_i = \pi_{i-1} \cdot p_{i-1,i} + \pi_{i+1} \cdot p_{i+1,i}, \quad \forall i = 1, 2, \dots, K-1 \text{ (3 points)}$$

Solving this gives $\pi_i = \binom{K}{i}\pi_0$, yielding $\pi_i = \binom{K}{i}2^{-K}$ (3 points).

(d) By (c), the Markov chain is irreducible. It is further obvious that it is periodic. It follows that

- X_n does not converge(2 points), because we notice that X_n must have the same parity with n(1 points).
- Let $z \sim \text{Binom}(K, 1/2)$, then $Y_n \to \sum_{i=1}^{K} i^2 \cdot \pi_i = \text{Var}(z) + \mathbb{E}[z]^2 = \frac{K+K^2}{4}$ almost surely (3 points, give 2 points if the convergence notion is wrong.).

5 Sisyphean Chain [3+8+8]

Throughout this problem, $p_0, p_1, \dots$ is a given sequence of numbers in the interval $[0,1]$.

a) Let $(F_n)_{n\geq 0}$ be independent Bernoulli trials, with $F_n \sim \text{Bernoulli}(1-p_n)$ for each $n \geq 0$. Let $N = \min\{n \geq 0 : F_n = 1\}$ be the time of the first success. Compute $\mathbb{E}[N]$ in terms of the given sequence $p_0, p_1, \dots$.

b) Let $(X_n)_{n\geq 0}$ be a Markov chain on state space $\{0,1,2,\dots\}$ with nonzero transition probabilities given by

$$P_{n,0} = 1 - p_n, \quad P_{n,n+1} = p_n, \quad n \geq 0.$$

Find necessary and sufficient conditions on $p_0, p_1, \dots$ for this chain to be irreducible and positive recurrent.

c) Assuming the conditions of part (b) hold, what is the stationary distribution for the chain $(X_n)_{n\geq 0}$ in terms of $p_0, p_1, \dots$?

---

a) We can use indicators or the tail sum formula. The latter gives

$$\mathbb{E}[N] = \sum_{n\geq 0} P(N > n) = \sum_{n\geq 0} P(F_0 = 0, \dots, F_n = 0) = \sum_{n\geq 0} \prod_{k=0}^{n} p_k.$$

b) For irreducibility, we need $p_n > 0$ for all $n \geq 0$. For positive recurrence, starting in state 0, we need expected return time to be finite. The expected return time is one plus what we computed in part (a), so necessary conditions are:

$$p_n > 0 \ \forall n \geq 0, \quad \text{and} \quad \sum_{n\geq 0} \prod_{k=0}^{n} p_k < \infty.$$

They are also sufficient. The first gives $0 \to n$, and with the second (that return time to zero is finite) implies $n \to 0$, so $0 \leftrightarrow n$ for all n; irreducibility follows. Positive recurrence (a class property) follows since the average round-trip time from state 0 is finite.

c) We know a SD exists and is unique, so we should solve $\pi P = \pi$, which reads as:

$$\sum_{n\geq 0} \pi_n (1 - p_n) = \pi_0, \quad \text{and} \quad \pi_n p_n = \pi_{n+1} \ \ \forall n \geq 0.$$

Note that iterating the second implies

$$\pi_n p_n = \pi_{n-1} p_{n-1} p_n = \cdots = \pi_0 \prod_{k=0}^{n} p_k \quad \Rightarrow \quad \pi_n = \pi_0 \prod_{k=0}^{n-1} p_k.$$

Now, π_0 can be computed as the inverse of the expected return time: $\pi_0 = \frac{1}{1+\sum_{n\geq 0}\prod_{k=0}^{n} p_k}$.