EE127A
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EE 127A Final: Solutions

NAME: **SID:**

The exam lasts 3 hours. The maximum number of points is 50. Notes are not allowed except for a two-sided cheat sheet of regular format.

This booklet is 17 pages total, with extra blank spaces allotted throughout, and 2 blank pages at the end, left for you to write your answers.

There are 8 separate problems, ordered by topics consistent with the course outline. All the questions in this exam can be solved independently of each other.

1. (*6 points*, Topic: symmetric eigenvalues.) We consider a vector $u \in \mathbf{R}^n$, which is normalized ($\|u\|_2 = 1$), and the associated symmetric matrix $P = I - uu^T$.

   (a) (*2 points*) Show that P is positive semi-definite. *Hint:* use the Cauchy-Schwartz inequality.

   (b) (*2 points*) Show that the eigenvalues of P are 1 and 0.

   (c) (*2 points*) Using the fact that, starting from a vector u, you can find (via QR for example) $n-1$ vectors $u_2, \ldots, u_n$ such that $(u, u_2, \ldots, u_n)$ forms an orthonormal basis of $\mathbf{R}^n$, find an eigenvalue decomposition of P.

   Solution:

   (a) For every $x \in \mathbf{R}^n$, we have $x^T P x = x^T x - (x^T u)^2$. From the Cauchy-Schwartz inequality, we have

   $$|x^T u| \leq \|x\|_2 \cdot \|u\|_2 = \|x\|_2,$$

   which implies $x^T P x \geq 0$.

   (b) A non-zero vector x is an eigenvector corresponding to the eigenvalue λ if and only if

   $$\lambda x = Px = x - (x^T u)u.$$

   Thus the condition

   $$(1 - \lambda)x = (x^T u)u, \quad x \neq 0$$

   fully characterizes eigenvalue/eigenvector pairs (x, λ).

   The above equation has a solution for $\lambda = 1$, in which case any non-zero x such that $x^T u = 0$ solves it. Thus $\lambda = 1$ is an eigenvalue. If $\lambda \neq 1$, then $x = \alpha u$, with $\alpha = (x^T u)/(1 - \lambda) = \alpha/(1 - \lambda)$. Thus, $\lambda = 0$ is an eigenvalue, with eigenvector u.

   (c) We use the QR decomposition to find $n-1$ vectors $u_2, \ldots, u_n$ such that the collection $u, u_2, \ldots, u_n$ forms an orthonormal basis of $\mathbf{R}^n$. Then from the expression of P, we obtain that $Pu = 0$, and $Pu_i = u_i$, $i = 2, \ldots, n$. Thus $U := [u, u_2, \ldots, u_n]$ is an orthonormal matrix of eigenvectors of P, corresponding to the $n \times n$ diagonal matrix of eigenvalues $\Lambda = \mathbf{diag}(0, 1, \ldots, 1)$. We can write the eigenvalue decomposition as

   $$P = U \Lambda U^T.$$

   The above can be obtained directly from the fact that

   $$I = UU^T = uu^T + u_2 u_2^T + \ldots + u_n u_n^T,$$

   so that

   $$P = I - uu^T = u_2 u_2^T + \ldots + u_n u_n^T = U \Lambda U^T.$$

2. (*8 points*, Topic: projections.) We consider a collection of m news articles that are represented as m points x_i, $i = 1, \ldots, m$ in $\mathbf{R}^n$, with a bag-of-words method. Here n is the total number of words in a given dictionary, and the j-th element in x_i contains the number of times word j appears in article i. We are given two additional articles, called the *origin* and *destination* articles, represented by two vectors x_0 and x_d in $\mathbf{R}^n$. We would like to find a small number $k << m$ of articles that are in some way connecting the origin and destination articles (think of trying to find a sequence of articles that connect the beginning and the end of the coverage of a certain topic). To this end, we will find the k points that are closest to the *line* $\mathcal{L}$ (not the segment) passing through x_0 and x_d.

   (a) (*2 points*) Show that to simplify, and without loss of generality, we can reduce the problem to the case when $x_0 = 0$ and x_d is normalized ($\|x_d\|_2 = 1$).

   (b) (*2 points*) For a given point $x \in \mathbf{R}^n$, find the point on the line $\mathcal{L}$ that is closest, in Euclidean norm, to x.

   (c) (*2 points*) Show that the distance D from x to its projection on the line satisfies $D^2 = x^T P x$, where $P = I - x_d x_d^T$.

   (d) (*2 points*) Explain how you can find a set of k articles that connect x_0 and x_d, in the sense defined above.

   Solution:

   (a) We first replace x_i, x_d with $x_i - x_0$ and $x_d - x_0$ respectively. This allows to replace x_0 with 0. Then we can normalize all the vectors involved, dividing them by x_d. Note that the two operations involved (translation and scaling) do not change the geometry of the problem.

   (b) We must solve the projection problem

   $$D^2 := \min_t \ \|tx_d - x\|_2^2.$$

   Using the fact that x_d is normalized, we have

   $$\|tx_d - x\|_2^2 = t^2 - 2t(x^T x_d) + x^T x = (t - x^T x_d)^2 + x^T x - (x^T x_d)^2.$$

   The minimum is attained with $t^* = x^T x_d$, so that the projection of x on the line passing through 0 and x_d is $(x^T x_d) x_d$.

   (c) The optimal value of the problem is

   $$D^2 = x^T x - (x^T x_d)^2 = x^T P x,$$

   where $P = I - x_d x_d^T$, as claimed.

   (d) To connect x_0 and x_d, we simply compute the quantities

   $$D_i^2 := x_i^T P x_i, \quad i = 1, \ldots, m,$$

   and select the k smallest.

3. (*6 points*, Topic: SVD.) An exam with m questions is given to n students. The instructor collects all the grades in a $n \times m$ matrix G, with G_{ij} the grade obtained by student i on question j. We would like to assign a difficulty score to each question, based on the available data.

   (a) (*2 points*) Assume that the grade matrix G is well approximated by a rank-one matrix sq^T, with $s \in \mathbf{R}^n$ and $q \in \mathbf{R}^m$. (You may assume that both s, q have non-negative components.) Explain how to use the approximation to assign a difficulty score to each question.

   (b) (*2 points*) What is the interpretation of vector s?

   (c) (*2 points*) How would you compute a rank-one approximation to G? State precisely your answer in terms of the SVD of G.

   Solution:

   (a) Assume that $G = sq^T$, with $s \in \mathbf{R}^n$ and $q \in \mathbf{R}^m$. This means that the grade of the i-th student on question j is $s_i q_j$. Hence all the students score according to the same profile $q = (q_1, \ldots, q_m)$, up to a student-dependent scaling given by s. The vector q can be interpreted as the difficulty score of each question.

   (b) The vector s gives the relative strengths of the students: a student having a high value of s scores better than one with a low value.

   (c) To approximate G by a rank-one vector, we simply compute the SVD of G and select the singular vectors corresponding to the largest singular value. Precisely, we set $s = \sqrt{\sigma_1} u_1$, $q = \sqrt{\sigma_1} v_1$, where u_1, v_1 are the first columns of the matrices U, V in the SVD of $G = U \Sigma V^T$, and σ_1 is the largest singular value, appearing in the $(1, 1)$ position of the diagonal matrix of singular values Σ. (Note that the overall scale factor $\sqrt{\sigma_1}$ is irrelevant, as we are only interested in relative values of s, q.)

4. (*6 points*, Topic: convexity.) In this problem we examine the convexity of various functions of a n-vector x.

   (a) (*3 points*) For a given $k \in \{1, \ldots, n\}$ we define the function $s_k : \mathbf{R}^n \to \mathbf{R}$ with values given by
   $$s_k(x) = \sum_{i=1}^{k} x_{[i]},$$
   where $x_{[i]}$ is the i-th largest component of x. Show that s_k is convex. *Hint:* express s_k as the maximum of linear functions. You can try with $n = 3$, $k = 2$ first.

   (b) (*3 points*) Assume $n = 2k - 1$ is odd. Consider the average absolute deviation from the median of the components of x, which is the function $\phi : \mathbf{R}^n \to \mathbf{R}$ with values given by
   $$\phi(x) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \mathrm{med}(x)|,$$
   where $\mathrm{med}(x) = x_{[k]}$ denotes the *median* of the components of x. Show that ϕ is convex. *Hint:* express ϕ in terms of s_k, s_{k-1} and the sum of the components of x.

Solution:

(a) For $n = 3$, $k = 2$, we have
$$s_2(x) = \max(x_1 + x_2, x_2 + x_3, x_3 + x_1).$$
More generally:
$$s_k(x) = \max_{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k} x_{i_1} + \ldots + x_{i_k}.$$
This shows that s_k is the point-wise maximum of linear function, hence it is convex.

(b) We have, for $n = 2k - 1$ odd:
$$\phi(x) = \sum_{i=1}^{k-1}(x_{[i]} - x_{[k]}) + \sum_{i=k+1}^{n}(x_{[k]} - x_{[i]}) = \sum_{i=1}^{k-1} x_{[i]} - \sum_{i=k+1}^{n} x_{[i]} = s_{k-1}(x) + s_k(x) - \sum_{i=1}^{n} x_i,$$
which shows that ϕ is convex.

5. (*6 points*, Topic: sphere packing.) We consider the problem of packing a given number m of spheres in a box of minimal area. The spheres have a given radius r_i, and the problem is to determine the location of the centers x_i, $i = 1, \ldots, m$. The constraints in this problem are that the spheres should not overlap, and should be contained in a square of center 0 and half-size R. The objective is to minimize the area of the containing box.

   (a) (*2 points*) Show that two spheres of radius r_1, r_2 and centers x_1, x_2 respectively do not intersect if and only if $\|x_1 - x_2\|_2$ exceeds a certain number, which you will determine.

   (b) (*2 points*) Formulate the sphere packing problem as an optimization problem in variables $x_1, \ldots, x_m$, and R.

   (c) (*2 points*) Is the formulation you have found convex? If not, state precisely why. Otherwise, state which acronym seen in the course (LP, QP, SOCP, SDP) applies, if possible.

Solution:

(a) The non-overlap condition is

$$\|x_1 - x_2\|_2 \geq r_1 + r_2.$$

(b) A sphere of center x and radius r is contained in the box of center 0 and half-size R if and only if the center is inside the box of center 0 and half-size $R - r$, that is: $\|x\|_\infty \leq R - r$.

Hence the sphere packing problem writes

$$\min_{R, x_1, \ldots, x_m} R \ : \ \begin{array}{l} \|x_i - x_i\|_2 \geq r_i + r_j, \ \ 1 \leq i, j \leq m, \\ \|x_i\|_\infty + r_i \leq R, \ li = 1, \ldots, m. \end{array}$$

(c) The problem, as formulated above, is not convex, since the non-overlap constraints involve bounding norms from *below*.

6. (*8 points*, Topic: LP.) When a user goes to a website, one of a set of n ads, labeled $1, \ldots, n$, is displayed. This is called an impression. We divide some time interval (say, one day) into T periods, labeled $t = 1, \ldots, T$. Let $N_{it} \geq 0$ denote the number of impressions in period t for which we display ad i. In period t there will be a total of $I_t > 0$ impressions, so we must have $\sum_{i=1}^{n} N_{it} = I_t$, for $t = 1, \ldots, T$. (The numbers I_t might be known from past history.) You can treat all these integer numbers as real. (This is justified since they are typically very large.)

   The revenue for displaying ad i in period t is $R_{it} \geq 0$ per impression. (This might come from click-through payments, for example.) The total revenue is $\sum_{t=1}^{T} \sum_{i=1}^{n} R_{it} N_{it}$. To maximize revenue, we would simply display the ad with the highest revenue per impression, and no other, in each display period.

   We also have in place a set of m contracts that require us to display certain numbers of ads, or mixes of ads (say, associated with the products of one company), over certain periods, with a penalty for any shortfalls. Contract j is characterized by a set of ads $\mathcal{A}_j \subseteq \{1, \ldots, n\}$ (while it does not affect the math, these are often disjoint), a set of periods $\mathcal{T}_j \subseteq \{1, \ldots, T\}$, a target number of impressions $q_j \geq 0$, and a shortfall penalty rate $p_j > 0$. The shortfall s_j for contract j is

$$s_j = \max\left(0, q_j - \sum_{t \in \mathcal{T}_j} \sum_{i \in \mathcal{A}_j} N_{it}\right).$$

   (This is the number of impressions by which we fall short of the target value q_j.) Our contracts require a total penalty payment equal to $\sum_{j=1}^{m} p_j s_j$. Our net profit is the total revenue minus the total penalty payment.

   Explain how to find the display numbers N_{it} that maximize net profit via linear programming. The data in this problem are $R \in \mathbf{R}^{n \times T}$, $I \in \mathbf{R}^{T}$ (here I is the vector of impressions, not the identity matrix), and the contract data $\mathcal{A}_j$, $\mathcal{T}_j$, q_j and p_j, $j = 1, \ldots, m$. Make sure to state precisely what the variables and constraints are.

   Solution: The problem reads as an LP with variables the matrix $N \in \mathbf{R}^{n \times T}$ and vector $s \in \mathbf{R}^m$:

$$\begin{aligned}
\max_{N,s} \quad & \sum_{t=1}^{T} \sum_{I=1}^{n} R_{it} N_{it} - \sum_{j=1}^{m} p_j s_j \\
\text{s.t.} \quad & s_j \geq q_j - \sum_{t \in \mathcal{T}_j} \sum_{i \in \mathcal{A}_j} N_{it}, \quad j = 1, \ldots, m, \\
& \sum_{i=1}^{n} N_{it} = I_t, \quad t = 1, \ldots, T, \\
& N \geq 0, \quad s \geq 0,
\end{aligned}$$

   where the last two inequalities are component-wise.

7. (*10 points*, Topic: QP.) We are given a fixed number of shares $\bar{s}$ of a single asset, to be purchased over time intervals $t = 1, \ldots, T$. We denote by s_t the amount of shares to be purchased at time t, and refer to the vector $s = (s_1, \ldots, s_T) \in \mathbf{R}^T$ as our sequence of trades, so that $s^T \mathbf{1} = \bar{s}$, where $\mathbf{1} \in \mathbf{R}^T$ is a vector of ones. We treat s as a real vector (not an integer vector), and do not allow short selling, that is, we impose the constraint $s \geq 0$. We denote by p_t the price of the asset at time t, and refer to $p = (p_1, \ldots, p_T)$ as the price vector. The *execution cost* associated with a given sequence of trades $s \in \mathbf{R}^T_+$ is then $\sum_{t=1}^T p_t s_t$.

   As we purchase s_t shares at each time t, $t = 1, \ldots, T$, the price p_t changes, not only due to (random) market dynamics, but also due to our purchases. A simple model for market impact dynamics is

   $$p_t = p_{t-1} + \alpha s_t + r_t, \quad t = 0, \ldots, T, \tag{1}$$

   where p_0, $\alpha > 0$ are model parameters, which we assume known. Here, the exogenous signal $r = (r_1, \ldots, r_T)$, which we also assume to be known for now, reflects the influence of the market as a whole on the price; for example, it may be derived from a simple (*e.g.*, auto-regressive) model for the SP 500 index. The market impact model above is simplistic as it does not guarantee positive prices, but we ignore that fact here.

   Our goal is to find the best sequence of trades s so as to minimize the execution cost, subject to the constraints on s.

   (a) (*2 points*) Show that we can write $p = A(\alpha s + r) + q$, where A a lower-triangular $T \times T$ matrix with 1's on the lower-triangular part, and $q \in \mathbf{R}^T$ is given.

   (b) (*4 points*) Write the problem with decision variables s, p, and including the constraint (1). In that form, is the problem convex? Justify your answer carefully.

   (c) (*4 points*) Write the problem as a QP in standard form. State precisely the variables and constraints. Make sure to check that the objective function is quadratic and convex in the variables of the problem. *Hint:* show that $q(s) := s^T A s = s^T Q s$, with $Q := (1/2)s^T(A + A^T)s$, and that $2Q - I$ is PSD, with I the $T \times T$ identity matrix.

   Solution:

   (a) We have

   $$p = A(\alpha s + r) + q,$$

   where

   $$A = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}, \quad q = \begin{pmatrix} p_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(b) The problem writes

$$\min_{s,p} \ p^T s \ : \ s \geq 0, \ \ s^T \mathbf{1} = \bar{s}, \ \ p = A(\alpha s + r) + q. \tag{2}$$

As such it is not a QP. To check this, we observe that the objective function, $f : (p, s) \to p^T s$, is quadratic, and contains no linear of constant terms (it is a "quadratic form"). Quadratic forms are convex if and only if they are non-negative everywhere. This is not true, as the special case $p = -s$ reveals.

(c) Problem (2) is not convex. However, after we eliminate p, the problem becomes convex.

Indeed, the execution costs are

$$s^T p = s^T (A(\alpha s + r) + q) = \alpha s^T A s + s^T (Ar + q).$$

Since $\alpha > 0$ it suffices to show that the quadratic function $q : \mathbf{R}^m \to \mathbf{R}$ with values $q(s) := s^T A s$ is convex. We use the hint:

$$q(s) = s^T A s = \frac{1}{2}(s^T A s + s^T A^T s) = s^T Q s,$$

With $Q := (1/2)(A + A^T)$. It is readily verified that the diagonal elements of $2Q$ are all 2's and off-diagonal elements are all 1's:

$$2Q = A + A^T = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} = I + \mathbf{1}\mathbf{1}^T.$$

Hence the form q is positive semi-definite:

$$\text{For every } s\text{: } q(s) = s^T Q s = \frac{1}{2}(s^T s + (\mathbf{1}^T s)^2) \geq 0.$$

Hence the matrix Q is positive semi-definite (PSD), and the associated quadratic function q is convex.

To summarize, our problem writes

$$\min_{s} \ \alpha s^T Q s + s^T (Ar + q) \ : \ s \geq 0, \ \ s^T \mathbf{1} = \bar{s}.$$

This is a QP (since $\alpha > 0$, Q PSD).