EE127A 9/28/11
L. El Ghaoui

Quiz 1: Solutions

1. Consider the matrix $A = uv^T$, with $u \in \mathbf{R}^n$, $v \in \mathbf{R}^m$.

   (a) Find the nullspace and range of A.

   (b) Explain how to compute an SVD of A.

   Solutions: We assume $u \neq 0$, $v \neq 0$ to avoid trivialities.

   (a) For the nullspace, the condition $Ax = 0$ is written $(v^T x)u = 0$. Since $u \neq 0$, we obtain $v^T x = 0$. That is, the nullspace is the hyperplane going through zero of vectors orthogonal to v.

   For the range, we look at the set of vectors of the form $(v^T x)u$, when x ranges the whole space $\mathbf{R}^n$. Clearly it is included in the line going through the origin with direction u, $\mathcal{L} := \{tu \ : \ t \in \mathbf{R}\}$. Now for any point in $\mathcal{L}$, say of the form tu with $t \in \mathbf{R}$, we can find x such that $t = v^T x$; for example $x = (t/v^T v)v$. Thus the nullspace equals to the line $\mathcal{L}$.

   (b) We can write $A = \sigma pq^T$, where $p = u/\|u\|_2$, $q = v/\|v\|_2$, $\sigma = \|u\|_2 \cdot \|v\|_2$. This is basically the SVD of A in short form. The full version would require to complete the vector p (resp. q) via orthogonalization to form an orthogonal matrix U that contains p as its first column. Similarly we form an orthogonal matrix V that contains q as its first column. Finally, we set $S = \mathbf{diag}(\sigma, 0, \ldots, 0)$ of size $n \times m$. We have

   $$USV^T = \sigma pq^T = A.$$

   This proves that the triple (U, S, V) is an SVD of A.

2. Consider the 2×2 matrix

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{2}{\sqrt{10}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

(a) What is an SVD of A? Express it as $A = USV^T$, with S the diagonal matrix of singular values ordered in decreasing fashion. Make sure to check all the properties required for U, S, V.

(b) Find the semi-axis lengths and principal axes of the ellipsoid

$$\mathcal{E}(A) = \left\{ Ax \ : x \in \mathbf{R}^2, \ \ \|x\|_2 \le 1 \right\}.$$

Hint: Use the SVD of A to show that every element of $\mathcal{E}(A)$ is of the form $y = U\bar{y}$ for some element $\bar{y}$ in $\mathcal{E}(S)$. That is, $\mathcal{E}(A) = \{U\bar{y} \ : \ \bar{y} \in \mathcal{E}(S)\}$. (In other words the matrix U maps $\mathcal{E}(S)$ into the set $\mathcal{E}(A)$.) Then analyze the geometry of the simpler set $\mathcal{E}(S)$.

(c) What is the set $\mathcal{E}(A)$ when we append a zero vector after the last column of A, that is A is replaced with $\tilde{A} = [A, 0] \in \mathbf{R}^{2\times 3}$?

(d) Same question when we append a row after the last row of A, that is, A is replaced with $\tilde{A} = [A^T, 0]^T \in \mathbf{R}^{3\times 2}$. Interpret geometrically your result.

Solution:

(a) We have

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = USV^T,$$

where $U = [u_1, u_2]$, $V = [v_1, v_2]$ and $S = \mathbf{diag}(\sigma_1, \sigma_2)$, with $\sigma_1 = 2$, $\sigma_2 = 1$, and

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The triplet (U, S, V) is an SVD of A, since S is diagonal with non-negative elements on the diagonal, and U, V are orthogonal matrices ($U^TU = V^TV = I_2$). To check this, we first check that the Euclidean norm of u_1, u_2, v_1, v_2 is one. (This is why we factored the term $\sqrt{10}$ into $\sqrt{2} \cdot \sqrt{5}$.) In addition, $u_1^T u_2 = v_1^T v_2 = 0$. Thus, U, V are orthogonal, as claimed.

(b) We have, for every x, $y := Ax = US(V^Tx)$ hence $y = U\bar{y}$, with $\bar{y} = S\bar{x}$ and $\bar{x} = V^Tx$. Since V is orthogonal, $\|\bar{x}\|_2 = \|x\|_2$. In fact, when x runs the unit Euclidean sphere, so does $\bar{x}$. Thus every element of $\mathcal{E}(A)$ is of the form $y = U\bar{y}$ for some element $\bar{y}$ in $\mathcal{E}(S)$. To analyze $\mathcal{E}(A)$ it suffices to analyze $\mathcal{E}(S)$ and then transform the points of the latter set via the mapping $\bar{y} \to U\bar{y}$.

Since

$$\mathcal{E}(S) = \left\{ \sigma_1 \bar{x}_1 e_1 + \sigma_1 \bar{x}_2 e_2 \ : \ \bar{x}_1^2 + \bar{x}_2^2 \le 1 \right\},$$

with e_1, e_2 the unit vectors, we have

$$\mathcal{E}(A) = \left\{ \sigma_1 \bar{x}_1 u_1 + \sigma_1 \bar{x}_2 u_2 \ : \ \bar{x}_1^2 + \bar{x}_2^2 \leq 1 \right\}.$$

In the coordinate system defined by the orthonormal basis (u_1, u_2) the set is an ellipsoid with semi-axis lengths (σ_1, σ_2), and principal axes given by the coordinate axes. In the original system the principal axes are u_1, u_2.

(c) When we append a zero column after the last column of A we are doing nothing to $\mathcal{E}(A)$. Indeed, the condition

$$y = Ax \text{ for some } x \in \mathbf{R}^2, \ \ \|x\|_2 \leq 1$$

is the same as

$$y = \begin{pmatrix} A & 0 \end{pmatrix} z \text{ for some } z \in \mathbf{R}^3, \ \ \|z\|_2 \leq 1.$$

Geometrically, the projection of a 3-dimensional unit sphere on the first two coordinates is the 2-dimensional unit sphere. Hence we loose nothing if the 2D sphere used to generate the points x is replaced by the projection of the 3D sphere.

(d) Here we append a row after the last row of A, replacing A with

$$\tilde{A} = \begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathbf{R}^{3 \times 2}.$$

The set $\mathcal{E}(\tilde{A})$ is the set of points of the form $(y, 0) \in \mathbf{R}^3$ where $y \in \mathcal{E}(A)$. This means that we are simply embedding the ellipsoid $\mathcal{E}(A)$ into a 3D space, instead of the original 2D one. The set $\mathcal{E}(\tilde{A})$ is now a degenerate (flat) ellipsoid in $\mathbf{R}^3$, entirely contained on the plane defined by the first two unit vectors in $\mathbf{R}^3$.