Optimization Models

EECS 127 / EECS 227AT

Laurent El Ghaoui

EECS department
UC Berkeley

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LECTURE 9

Linear Algebra Review

> *There is hardly any theory which is more elementary than linear algebra, in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices.*
>
> J. Dieudonné (1906—1992)

Vectors

- transpose
- scalar product
- l_p-norms ($p = 1, 2, \infty$), and (generalized) Cauchy-Schwarz inequalities
- Euclidean projection on a line
- orthogonalization (Gram-Schmidt procedure)
- linear and affine functions, gradient
- hyperplanes

Matrices

- transpose, trace
- matrix-vector and matrix-matrix products
- range, rank, nullspace
- matrix norms: Frobenius norm and maximum singular value norm
- special matrices: dyads, diagonal, triangular, orthonormal (unitary)
- QR decomposition

Symmetric matrices

- symmetric matrices and examples
- quadratic forms and functions, Hessian
- spectral theorem
- positive semidefinite matrices (a.k.a covariance matrices)
- ellipsoids
- PCA

SVD

- SVD theorem
- Low-rank approximation
- Link with PCA

Linear equations

- examples
- set of solutions
- over- and under-determined equations
- solution via SVD, QR

Least-squares

- ordinary least-squares and interpretations
- minimum-norm solution to under-determined set of linear equations
- regularized LS (with squared Euclidean norm)
- use of LS in prediction, cross-validation
- kernel LS
- AR models for time-series

Limits of the linear algebra approach

- Consider Ridge regression, and assume we have a-priori information on the coefficients. For instance, we know that they are positive.
- The problem becomes
$$\min_{a \geq 0} \|y - \Phi a\|_2^2 + \lambda \|a\|_2^2$$
- The constraint $a \geq 0$ makes the problem "a little harder." No longer we have a "closed-form," linear algebra solution.
- Another variation, using an ℓ_1 regularization term:
$$\min_{a} \|y - \Phi a\|_2^2 + \lambda \|a\|_1$$
  Again, no "linear algebra" solution ...
- We need new tools for attacking these (and many other) problems!
- It turns out that these problems can still be solved very efficiently, with a computational effort comparable to that of "linear algebra" solutions.

Limitations of linear algebra

In linear equations and least-squares:

- Inability to deal with inequality constraints (example with non-negative LS);
- Inability to deal with non-Euclidean norms in penalty terms;
- Inability to deal with non-Euclidean norms in the objective function.

Similar limitations appear in the context of low-rank approximations.