TODAY : SOCP!

① SOCP & duals (+dual norm application)

② a common SOCP mistake

③ casting problems as SOCPs

④ discussion of canonical problem forms



EECS 127/227AT Optimization Models in Engineering
Spring 2020

Discussion 11

**1. Dual norms and SOCP**

Consider the problem

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \underbrace{\|A\vec{x} - \vec{y}\|_1}_{\sum_i |(A\vec{x}-\vec{y})_i|} + \mu\|\vec{x}\|_2,$$

$\rightarrow p^* = \min_{\vec{x}} \vec{c}^\top \vec{x}$

s.t. $\| \quad \| \leq$ [affine]

where $A \in \mathbb{R}^{m\times n}$, $\vec{y} \in \mathbb{R}^m$, and $\mu > 0$.

(a) Express this (primal) problem in standard SOCP form.

slack variables!

$$\min_{\vec{x}, \vec{z}, t} \vec{z}^\top \vec{1} + \mu t$$

$$\text{s.t. } |(A\vec{x}-\vec{y})_i| \leq z_i, \; i=1,\dots,m$$

$$\|\vec{x}\|_2 \leq t$$

$$(A\vec{x}-\vec{y})_i \leq z_i$$

$$(A\vec{x}-\vec{y})_i \geq -z_i$$

$$|x| \leq t \;\Updownarrow\; x \leq t,\; x \geq -t$$

(b) Find a dual to the problem and express it in standard SOCP form.
*Hint: Recall that for every vector $\vec{z}$, the following dual norm equalities hold:*

$$\|\vec{z}\|_2 = \max_{\vec{u}\,:\,\|\vec{u}\|_2\leq 1} \vec{u}^\top \vec{z}, \quad \|\vec{z}\|_1 = \max_{\vec{u}\,:\,\|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z}.$$

$$p^* = \min_{\vec{x}} \underbrace{\|A\vec{x}-\vec{y}\|_1}_{\max_{\vec{u}:\|\vec{u}\|_\infty\leq 1} \vec{u}^\top(A\vec{x}-\vec{y})} + \mu \underbrace{\|\vec{x}\|_2}_{\max_{\vec{v}:\|\vec{v}\|_2\leq 1} \vec{v}^\top\vec{x}}$$

$$= \min_{\vec{x}} \max_{\substack{\vec{u}:\|\vec{u}\|_\infty\leq 1\\ \vec{v}:\|\vec{v}\|_2\leq 1}} \vec{u}^\top(A\vec{x}-\vec{y}) + \mu\vec{v}^\top\vec{x}$$

DUAL!

$$d^* = \max_{\substack{\vec{u}:\|\vec{u}\|_\infty\leq 1\\ \vec{v}:\|\vec{v}\|_2\leq 1}} \underbrace{\min_{\vec{x}} \vec{u}^\top(A\vec{x}-\vec{y}) + \mu\vec{v}^\top\vec{x}}$$

$$= \min_{\vec{x}} (\vec{u}^\top A + \mu\vec{v})\vec{x} - \vec{u}^\top\vec{y} = \begin{cases} -\vec{u}^\top\vec{y}, & A^\top\vec{u}+\mu\vec{v}=\vec{0} \\ -\infty, & \text{o.w.} \end{cases}$$

$$d^* = \max_{\vec{u},\vec{v}} -\vec{u}^T\vec{y}$$

$$\text{s.t. } A^T\vec{u} + \mu\vec{v} = \vec{0} \rightarrow A^T\vec{u} = -\mu\vec{v}$$

$$\|\vec{u}\|_\infty \le 1,\ \|\vec{v}\|_2 \le 1 \qquad \rightarrow \vec{V} = -\frac{A^T\vec{u}}{\mu}$$

$$= \max_{\vec{u}} -\vec{u}^T\vec{y} \checkmark$$

$$\text{s.t. } \|A^T\vec{u}\|_2 \le \mu \checkmark$$

$$\|\vec{u}\|_\infty \le 1 \rightarrow \max_i |u_i| \le 1$$

$$\Updownarrow$$

$$|u_i| \le 1$$

$$\Updownarrow$$

$$u_i \le 1$$

$$u_i \ge -1 \checkmark$$

(c) Assume strong duality holds[1] and that $m = 100$ and $n = 10^6$, i.e., $A$ is $100 \times 10^6$. Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

$$p^* = \min_{\vec{x},\vec{z},t} \vec{z}^T\vec{1} + \mu t$$

$$\text{s.t. } \left.\begin{array}{l}(A\vec{x}+\vec{y})_i \le z_i \\ (A\vec{x}+\vec{y})_i \ge -z_i\end{array}\right\} i=1,\dots,m$$

$$\|\vec{x}\|_2 \le t$$

$\rightarrow \sim 10^6$ variables

$\rightarrow$ 201 constraints

$$d^* = \max_{\vec{u}} -\vec{u}^T\vec{y}$$

$$\text{s.t. } \left.\begin{array}{l}u_i \le 1 \\ u_i \ge -1\end{array}\right\} i=1,\dots,m$$

$$\|A^T\vec{u}\|_2 \le \mu$$

$\rightarrow$ 100 variables

$\rightarrow$ 201 constraints

**2. Squaring SOCP constraints**

When considering a second-order cone (SOC) constraint, you might be tempted to square it to obtain a classical convex quadratic constraint. This problem explores why that might not always work, and how to introduce additional constraints to maintain equivalence and convexity.

(a) For $\vec{x} \in \mathbb{R}^2$, consider the constraint

$$x_1 - 2x_2 \ge \|\vec{x}\|_2. \rightarrow \text{SOC} \rightarrow \text{yes convex}$$

and its squared counterpart

$$(x_1 - 2x_2)^2 \ge \|\vec{x}\|_2^2. \rightarrow \text{not convex}$$

Are the two sets equivalent? Are they both convex?

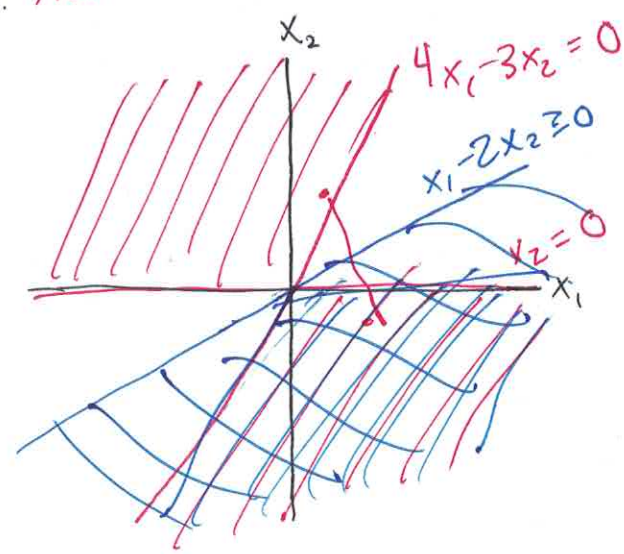
$$(x_1 - 2x_2)^2 \ge \|\vec{x}\|_2^2$$

$$x_1^2 - 4x_1x_2 + 4x_2^2 \ge x_1^2 + 2x_1x_2 + x_2^2$$

$$x_2(4x_1 - 3x_2) \le 0$$

$+ \quad -$

$- \quad +$




[1] In fact, you can show that strong duality holds using Sion's theorem, a generalization of the minimax theorem that is beyond the scope of this class.

(b) What additional constraint must be imposed alongside the squared constraint to enforce the same feasible set as the unsquared SOC constraint?

must add implicit constraint that $x_1 - 2x_2 \geq 0$

**3. Casting optimization problems as SOCPs**

Cast the following problem as an SOCP in its standard form:

$$\min_{\vec{x} \in \mathbb{R}^n} \sum_{i=1}^{p} \frac{\|F_i \vec{x} + \vec{g}_i\|_2^2}{\vec{a}_i^\top \vec{x} + b_i}$$
$$\text{s.t. } \vec{a}_i^\top \vec{x} + b_i > 0, \quad i = 1, \ldots, p,$$

where $F_i \in \mathbb{R}^{m \times n}$, $\vec{g}_i \in \mathbb{R}^m$, $\vec{a}_i \in \mathbb{R}^n$, and $b_i \in \mathbb{R}$, for $i = 1, \ldots, p$.

$$\min_{\vec{x}, \vec{t}} \sum_{i=1}^{p} t_i$$
$$\text{s.t. } \|F_i \vec{x} + g_i\|_2^2 \leq t_i(\vec{a}_i^\top \vec{x} + b_i)$$
$$\vec{a}_i^\top \vec{x} + b_i > 0 \quad i = 1, \ldots, p$$

$$\Longleftrightarrow$$

$$\min_{\vec{x}, \vec{t}} \sum_{i=1}^{p} t_i$$
$$\text{s.t. } \left\| \begin{bmatrix} 2(F_i \vec{x} + g_i) \\ t_i - \vec{a}_i^\top \vec{x} - b_i \end{bmatrix} \right\|_2 \leq t_i + \vec{a}_i^\top \vec{x} + b_i$$
$$i = 1, \ldots, p$$

EQUIVALENCE: Calafiore & El Ghaoui 10.2.3

**4. A review of standard problem formulations**

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) ***Linear programming (LP).***

i. Write the most general form of a linear program (LP) and list its defining attributes.

$$p^* = \min_{\vec{x}} \vec{c}^T \vec{x} + d$$
$$\text{s.t. } A_{eq}\vec{x} = \vec{b}_{eq}$$
$$A\vec{x} \le \vec{b}$$

OR

$$p^* = \min_{\vec{x}} \vec{c}^T \vec{x} + d$$
$$\text{s.t. } A_{eq}\vec{x} = \vec{b}_{eq}$$
$$\vec{x} \ge \vec{0}$$

- affine objective
- affine constraints

ii. Under what conditions is an LP convex?

always!

(b) ***Quadratic programming (QP).***

i. Write the most general form of a quadratic program (QP) and list its defining attributes.

$$p^* = \min_{\vec{x}} \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x} + d$$
$$\text{s.t. } A_{eq}\vec{x} = \vec{b}_{eq}$$
$$A\vec{x} \le \vec{b}$$

- quadratic objective
- affine constraints

ii. Under what conditions is a QP convex?

$H \succeq 0$ (PSD)

(c) ***Quadratically-constrained quadratic programming (QCQP).***

i. Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

$$p^* = \min_{\vec{x}} \vec{x}^T H_0 \vec{x} + 2\vec{c}^T\vec{x} + d$$
$$\text{s.t. } \vec{x}^T H_i \vec{x} + 2\vec{c}_i^T\vec{x} + d_i \le 0, \quad i = 1, \ldots, m$$
$$\vec{x}^T H_j \vec{x} + 2\vec{c}_j \vec{x} + d_j = 0, \quad j = 1, \ldots, p$$

- quadratic objective
- quadratic constraints

ii. Under what conditions is a QCQP convex?

$H_0 \succeq 0$ (convex objective)

$H_i \succeq 0 \ \forall i$ (convex ineq. constraints)

$H_j = 0 \ \forall j$ (affine eq. constraints)

(d) ***Second-order cone programming (SOCP).***

i. Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

$$p^* = \min_{\vec{x}} \vec{c}^T\vec{x}$$

$$\text{s.t. } \|A_i\vec{x} + \vec{b}_i\|_2 \le \vec{c}_i^T\vec{x} + d_i, \; i = 1, \ldots, m$$

- linear/affine objective
- SOC ("conic") constraints

ii. Under what conditions is an SOCP convex?

always!

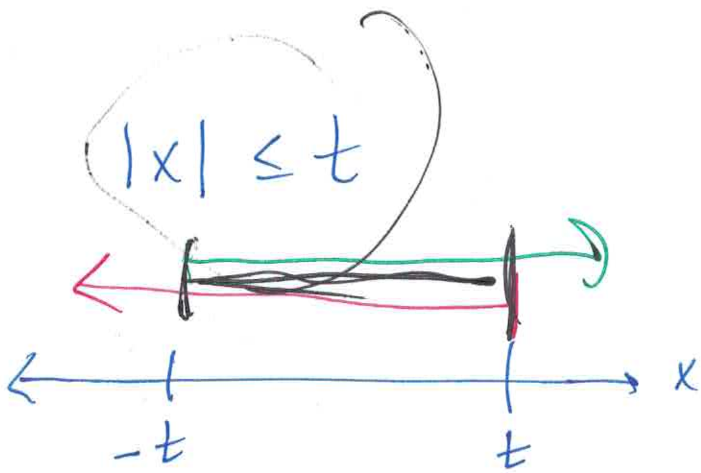
(e) ***Relationships.*** Recall that

$$LP \subset QP_{\text{convex}} \subset QCQP_{\text{convex}} \subset SOCP \subset \{\text{all convex programs}\},$$

← SDP ("semidefinite programs")

where $LP$ denotes the set of all linear programs, $QP_{\text{convex}}$ denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?

- L → R most → least efficient (generally)
- useful for:
  - → recognizing convexity
  - → taking advantage of existing algorithms/software




$\cancel{x \le t}$ $\quad x \le t$

$x \le -t$