Inverted pendulum: keep it stable without spending too much energy

EECS 127/227AT Optimization Models in Engineering
Spring 2020 Discussion 12

momentum

State $\vec{x}$

Control input $\vec{u}$

$\vec{x}^{t+1} = A\vec{x}^t + B\vec{u}^t$

You want to set $\vec{x}^t \to \vec{0}$ by controlling the system without paying to much on $\vec{u}$

1. LQR and least squares

In this question, we consider the time-dependent n-state m-input LQR problem

Cost if $x^t \neq 0$

Pay more if you are far from 0

Cost on the control

$$\min_{\vec{x}_t \in \mathbb{R}^n, \vec{u}_t \in \mathbb{R}^m} \sum_{t=0}^{T-1} \left(\vec{x}_t^\top Q_t \vec{x}_t + \vec{u}_t^\top R_t \vec{u}_t \right) + \vec{x}_T^\top Q_T \vec{x}_T$$

$(\vec{x}_t - \vec{x}_e)^\top Q_t (\vec{x}_t - \vec{x}_e)$

$$\text{s.t. } \vec{x}_{t+1} = A\vec{x}_t + B\vec{u}_t, \; t = 1, \ldots, T$$

$$\vec{x}_0 = \vec{x}_{\text{init}},$$

Dynamical system

where $Q_t = Q_t^\top \succeq 0$ for all $t = 0, \ldots, T$ and $R_t = R_t^\top \succeq 0$ for all $t = 0, \ldots, T-1$. Note that this is a minor extension of the standard LQR formulation explored in class — we allow the cost associated with each state $\vec{x}_t$ and input $\vec{u}_t$ to vary by time step. For clarity, note also that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q_t \in \mathbb{R}^{n \times n}$ for all $t = 0, \ldots, T$, and $R_t \in \mathbb{R}^{m \times m}$ for all $t = 0, \ldots, T-1$.

$\Leftrightarrow Q_t \succeq 0$

notation

In this problem, we reformulate this calculation as a least squares problem, examine its properties, and compare this solution strategy with others shown in class.

PSD matrices are always symmetric

(a) ***Concatenating variables of interest.*** We first make our formulation more concise by concatenating our states and inputs into single vectors and computing the associated matrices.

i. Define full state vector $\vec{x}$ and input vector $\vec{u}$ as follows:

$$\vec{x} = \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \\ \vdots \\ \vec{x}_T \end{bmatrix} \in \mathbb{R}^{n(T+1)}, \qquad \vec{u} = \begin{bmatrix} \vec{u}_0 \\ \vec{u}_1 \\ \vdots \\ \vec{u}_{T-1} \end{bmatrix} \in \mathbb{R}^{mT}.$$

Change of variable

Linear algebra is useful

Show that we can rewrite our LQR objective function as

$$\boxed{\vec{x}^\top Q \vec{x} + \vec{u}^\top R \vec{u}}$$

for some matrices $Q \in \mathbb{R}^{n(T+1) \times n(T+1)}$ and $R \in \mathbb{R}^{mT \times mT}$, which you determine.

$$Q = \begin{bmatrix} Q_0 & & 0 \\ & Q_1 & \\ 0 & \ddots & Q_T \end{bmatrix}. \quad R = \begin{bmatrix} R_0 & & 0 \\ & R_1 & \\ 0 & \ddots & R_{T-1} \end{bmatrix} \quad \vec{x} = \begin{bmatrix} \vec{x}_0 \\ \vdots \\ \vec{x}_T \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \vec{u}_0 \\ \vdots \\ \vec{u}_{T-1} \end{bmatrix}$$

$$\boxed{\left(\sum_{t=0}^{T-1} \vec{x}_t^\top Q_t \vec{x}_t + \vec{u}_t R_t \vec{u}_t \right) + \vec{x}_T Q_T \vec{x}_T = \vec{x} Q \vec{x} + \vec{u}^\top R \vec{u}}$$

ii. Show that we can reformulate our constraints (i.e., dynamics) as

$$\boxed{\vec{x} = G\vec{u} + H\vec{x}_{\text{init}}}$$

for some matrices $G \in \mathbb{R}^{n(T+1)\times mT}$ and $H \in \mathbb{R}^{n(T+1)\times n}$, which you determine.

$$\vec{x}_{t+1} = A\vec{x}_t + B\vec{u}_t$$

$$\vec{x}_0 = \vec{x}_{init}$$

$$\vec{x}_1 = A\vec{x}_{init} + B\vec{u}_0$$

$$\vec{x}_2 = A\vec{x}_1 + B\vec{u}_1 = A(A\vec{x}_{init} + B\vec{u}_0) + B\vec{u}_1 = A^2\vec{x}_{init} + AB\vec{u}_0 + B\vec{u}_1$$

$$\vdots$$

$$\vec{x}_T = A^T\vec{x}_{init} + \sum_{s=0}^{T-1} A^{T-s-1}B\vec{u}_s$$

$$\begin{bmatrix} \vec{x}_0 \\ \vdots \\ \vec{x}_T \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \cdots & & 0 \\ B & 0 & & \\ AB & B & \ddots & \\ A^{T-1}B & \cdots & AB & B \end{bmatrix}}_{G}\vec{u} + \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^T \end{bmatrix}}_{H}\vec{x}_{init}$$

$$\vec{x} = G\vec{u} + H\vec{x}_{init}$$

(b) ***Formulating the least squares problem.*** We have now reduced our LQR problem to

x is fully determined by u

Control

$$P = \boxed{\min_{\vec{x}\in\mathbb{R}^{n(T+1)},\,\vec{u}\in\mathbb{R}^{mT}} \vec{x}^\top Q\vec{x} + \vec{u}^\top R\vec{u} \quad \text{s.t. } \vec{x} = G\vec{u} + H\vec{x}_{\text{init}}.}$$

Dynamics

linear algebra calculus

Rewrite this optimization as an unconstrained least squares problem over $\vec{u}$.

$$P = \min_{\vec{u}\in\mathbb{R}^{mT}} \underbrace{(G\vec{u} + H\vec{x}_{init})^\top Q (G\vec{u} + H\vec{x}_{init})}_{\text{Cost on } x} + \underbrace{\vec{u}^\top R\vec{u}}_{\text{Cost on } u}$$

Cost on u how much energy you can use

Cost on ~~your objective line~~ for x

$$\underbrace{x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2}_{} = \left\| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{bmatrix} \right\|^2$$

$$\|Q^{\frac{1}{2}}(G\vec{u} + H\vec{x}_{init})\|_2^2 + \|R^{\frac{1}{2}}\vec{u}\|_2^2$$

$$= \left\| \begin{bmatrix} Q^{\frac{1}{2}}(G\vec{u}) \\ R^{\frac{1}{2}}\vec{u} \end{bmatrix} + \begin{bmatrix} H\vec{x}_{init} \\ 0 \end{bmatrix} \right\|_2^2 = \left\| \underbrace{\begin{pmatrix} Q^{1/2}G \\ R^{1/2} \end{pmatrix}}_{A}\vec{u} + \underbrace{\begin{pmatrix} H \\ 0 \end{pmatrix}\vec{x}_{init}}_{-\vec{b}} \right\|_2^2$$

$$\rightarrow \|A\vec{u} - \vec{b}\|_2^2 \quad \text{Least-square}$$

(c) ***Analysis.*** We now examine the practicality of using least squares to solve the LQR problem. *Note: This section is meant to provide intuition, not rigorous complexity analysis, and is presented primarily to illustrate the practical utility of different LQR methods. Do not feel obligated to understand the arguments below in detail.*

Recall from lecture that the LQR problem can also be solved via the following recursive procedure:

LS is a technique to solve LQR

(1) Set $P_T = Q_T$, then solve iteratively backward in time for "helper" matrices $P_{T-1}, \dots, P_0$ via Riccati equation

$$P_t = A^\top (I + P_{t+1} B R_t^{-1} B^\top)^{-1} P_{t+1} A + Q_t$$

(2) Solve iteratively forward in time for optimal $\vec{x}_1, \dots, \vec{x}_T$ and $\vec{u}, \dots, \vec{u}_{T-1}$ via

What is best between LS and recursive procedure?

$$\vec{u}_t = -R_t^{-1} B^\top (I + P_{t+1} B R_t^{-1} B^\top)^{-1} P_{t+1} A \vec{x}_t$$
$$\vec{x}_{t+1} = A\vec{x}_t + B\vec{u}_t$$

We will compare this strategy with the least squares formulation developed above.

i. Suppose $n = 2$, $m = 2$, and $T = 10,000$, i.e., we want to solve for the optimal control of a 2-state, 2-input system over a horizon of 10,000 time steps. Which solution method would you use? *Hint: Over long time horizons, computational efficiency is a major concern.*

LS takes more time than recursive solution

→ See solutions

linear constraints

ii. Suppose $n = 2$, $m = 2$, and $T = 100$, and we want to impose constraints on the control values $\vec{u}$ (e.g., each element of the $\vec{u}$ vector must remain between ± 10 units).[1] Which of these formulations might you choose to incorporate such constraints?

Add constraints on $\vec{u}$: Recursive approach does not work anymore → LS would still work

with linear constraints = QP

[1]Constraints like this are common in control problems; our motors/actuators usually can't provide infinite power!

2. Can we use slack variables?

So far, we've presented slack variables as a method of converting optimization problems to a desired form, and it may seem like we can always use them. In this question, we take a more nuanced look at when slack variables are helpful and when they are not.

For each of the following functions, consider the unconstrained optimization problem

$$p_j^* = \min_{\vec{x}\in\mathbb{R}^n} f_j(\vec{x})$$

If possible, reformulate each problem as an SOCP using slack variables. If not possible, explain why.[2]

EE 227B EECS227C Boyd book

Hw 11

(a) $f_1(\vec{x}) = \|A\vec{x} - \vec{y}\|_2 + \|\vec{x}\|_1$

$\min_{\vec x,t}\max_{\lambda \geq 0} t + \lambda(f_i(\vec x) - t)$ true if at optimal point constraint is active

$\min_{\vec x} f_1(\vec x) = \min_{\vec x, t} t$ s.t. $t = f_1(\vec x)$ Might be not convex

? $\min_{\vec x, t} t$ s.t. $t \geq f_1(\vec x)$ Convex if f_1 is convex

Showing constraint is active

① Contradiction: Assume not active ⇒ We can find a better optimal point ⇒ Constraint should be active at optimal

$\min_{\vec x, t}\max_{\mu} t + \mu(f_1(\vec x) - t)$ if you can show that $\mu \geq 0$ → Equality can be replaced to inequality

② Dual variable

$\min \|A\vec x - y\|_2 + \|\vec x\|_1$

$= \min \|A\vec x - y\|_2 + \sum_{i=1}^n |x_i|$

$= \min_{t, v_i, \vec x} t + \sum_{i=1}^n v_i$, $t \geq \|A\vec x - y\|_2$, $v_i \geq |x_i|$ $= \min t + \sum_{i=1}^n v_i$, $t \geq \|A\vec x - y\|_2$, $v_i \geq |x_i| = \max\{x_i, -x_i\}$

$v_i \geq -x_i$, $v_i \geq x_i$

→ SOCP

$\min t + \sum_{i=1}^n v_i$

$t \geq \|A\vec x - \vec y\|_2$

$v_i \geq -x_i \quad \forall i$

$v_i \geq x_i$

(b) $f_2(\vec{x}) = \|A\vec{x} - \vec{y}\|_2 - \|\vec{x}\|_1$

$f_2(\vec x) = \|A\vec x - y\|_2 - \|\vec x\|_1$

$\min_{\vec x, t, v_i} t - \sum_{i=1}^n v_i = t + \sum_{i=1}^n -v_i$

$t = \|A\vec x - \vec y\|_2$

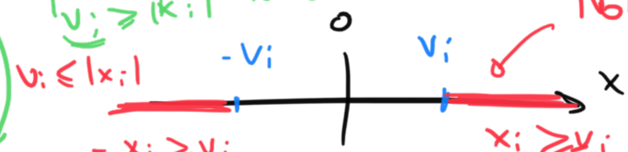
$v_i = |x_i|$

$-v_i = -|x_i|$

$-v_i \geq -|x_i| \to v_i \leq |x_i|$

$v_i \geq |x_i|$ is not a good relaxation of $v_i = |x_i|$

Not Convex



$\{x, |x_i| \geq v_i\}$

$\min t - \sum_{i=1}^n v_i$

$t \geq \|A\vec x - \vec y\|_2$

$\boxed{v_i \leq |x_i|}$ ← Not CONVEX

or easier to solve

Think if you can relax the equality to inequality without changing the solution (i.e. constraint will be active at optimal point) → Is the problem convex?

[2]You might notice that this problem is similar to Homework 11 problem 5, particularly part (c). Here, we're using this formulation to explore slack variables more precisely.