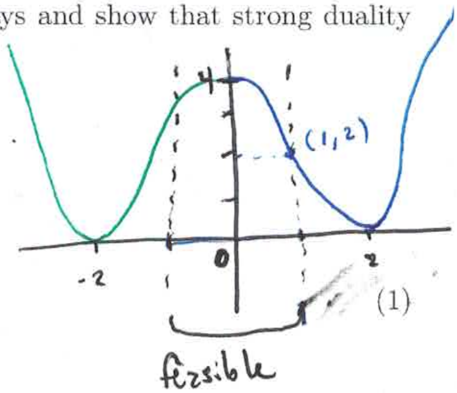
EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 9

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}.$$



(a) Consider the minimization problem

$$p^* = \inf_{x \in \mathbb{R}} f_0(x) \quad \text{s.t. } -1 \leq x, \; x \leq 1. \tag{1}$$

i. Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

$x = \pm\infty, \pm 2, 0, \pm 1$

$p^* = 2$ attained at $x \in \{-1, 1\}$.

ii. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda_1, \lambda_2 \geq 0} g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min\left\{g_1(\vec{\lambda}), g_2(\vec{\lambda})\right\},$$

with

$$g_1(\vec{\lambda}) = \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1) \quad h_1(x, \lambda)$$
$$g_2(\vec{\lambda}) = \inf_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1). \quad h_2(x, \lambda)$$

$$g(\lambda) = \inf \underbrace{\mathcal{L}(x, \lambda)}_{f_0(x) + \lambda_1(-1-x) + \lambda_2(x-1)}$$

$$= \min\left(\inf_{x \geq 0} \mathcal{L}(x, \lambda), \inf_{x < 0} \mathcal{L}(x, \lambda)\right)$$

iii. Next, show that

$$g_1(\vec{\lambda}) \leq -3\lambda_1 + \lambda_2$$
$$g_2(\vec{\lambda}) \leq \lambda_1 - 3\lambda_2.$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

$g_1(\lambda) \leq h(x, \lambda)$ at any feasible x.

$x = 2 \Rightarrow g_1(\lambda) \leq -3\lambda_1 + \lambda_2$

$x = -2 \Rightarrow g_2(\lambda) \leq \lambda_1 - 3\lambda_2$

iv. Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

~~$d^* = \max \{ g_1$~~

$$g(\vec{0}) = \min \{ g_1(\vec{0}), g_2(\vec{0}) \}$$
$$= \min \{ \inf_{x \geq 0} x^3 - 3x^2 + 4, \inf_{x < 0} -x^3 - 3x^2 + 4 \}$$
$$= \min \{ 0, 0 \} = 0 \Rightarrow d^* = \cancel{\sup_{\lambda} 0} = 0$$

more math

v. Does strong duality hold?

$p^* = 2$

$d^* = 0$

$\Rightarrow$ NO!

(b) Now, consider a problem equivalent to the minimization in (1):

$$p^* = \inf_{x \in \mathbb{R}} f_0(x) \quad \text{s.t. } x^2 \leq 1 \tag{2}$$

Observe that $p^* = 2$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part (a).

i. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda \geq 0} g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$g_1(\lambda) = \inf_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \quad h(x, \lambda)$$
$$g_2(\lambda) = \inf_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).$$

ii. Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3-\lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

skip in sol'n

$x = 0, +\infty, \nabla_x h(x, \lambda) = 0 \Rightarrow 3x^2 - 2(3-\lambda)x = 0$

$\Rightarrow x = 0$ or $x = \frac{2}{3}(3-\lambda)$

iii. Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

iv. Does strong duality hold?

$d^* = 2 = p^* \Rightarrow$ YES strong duality!

If you have a nonconvex problem you want to solve, try different equivalent specifications of your constraints.

$\min_{x} c^T x$

s.t. $Ax \leq b$

$x \leq 0$? ←

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$

2. Linear programming

Express the following problems as LPs.

(a) → slack variables!

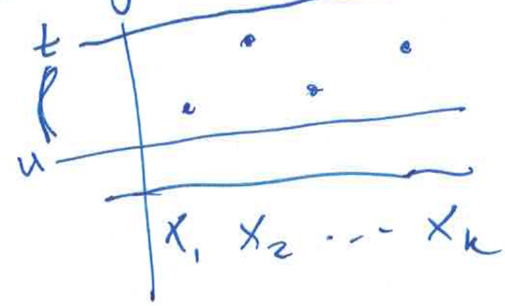
$$\min_{\vec{x}\in\mathbb{R}^k} \left[\max_{i=1,\ldots,k} x_i - \min_{j=1,\ldots,k} x_j\right]$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$

$$\min_{\vec{x}, t, u} [t - u]$$

s.t. $t \geq \max_i x_i \Leftrightarrow t \geq x_i \ \forall i$

$u \leq \min_j x_j \Leftrightarrow u \leq x_j \ \forall j$ ✓

$Ax = b$



(b)

$$\min_{\vec{x}\in\mathbb{R}^k} \sum_{i=1}^{k} |x_i|$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$

$$\min_{\vec{x}, \vec{t}} \sum_{i=1}^{k} t_i$$

s.t. $A\vec{x} = \vec{b}$

$t_i \geq |x_i|$

$x_i \leq t_i \quad x_i \geq -t_i$

$$\min_{\vec{x}, \vec{t}} \sum t_i \Leftrightarrow 1^T \vec{t}$$

s.t. $A\vec{x} = \vec{b}$

$x_i \leq t_i, \ x_i \geq -t_i \ \forall i$ //