EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 9

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

One variable in $\mathbb{R}$

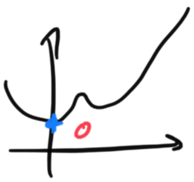
$$\underline{f_0(x)} \doteq \underbrace{\begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}} .$$

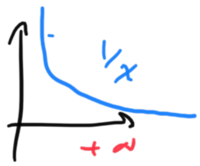
Not convex

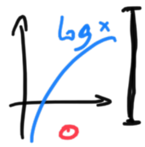
Taylor expansion of function

Intermediate value theorem

(a) Consider the minimization problem







$$p^* = \inf_{x \in \mathbb{R}} f_0(x) \quad \text{s.t.} \quad \boxed{-1 \leq x, \quad x \leq 1} \tag{1}$$

if $f_0(x)$ is differentiable and continuous optimal point is boundary or $\nabla f_0(x) = 0$

i. Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

$f_0(x)$ is continuous as polynomial in 0 $f(0) = 4$

$$\frac{df_0}{dx}(x) = \begin{cases} 3x^2 - 6x & x \geq 0 \\ -3x^2 - 6x & x < 0 \end{cases}$$

Also continuous, so $f_0 \in C^1(\mathbb{R}, \mathbb{R})$

$\frac{df_0}{dx}(x) = 0 \Rightarrow x = 1, 0, -1$ $\quad f(0) = 4$, $f(\pm 1) = 2$

not a minimum

Boundaries $\pm\infty \Rightarrow f(\pm\infty) = +\infty$

$p^* = 2 \quad \operatorname{argmin} f(x) = \pm 1$

ii. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda_1, \lambda_2 \geq 0} g(\vec{\lambda}),$$

where

$$\underline{g(\vec{\lambda}) = \min\left\{g_1(\vec{\lambda}), g_2(\vec{\lambda})\right\},}$$

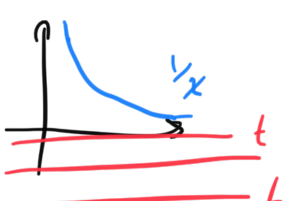
with

$$g_1(\vec{\lambda}) = \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1)$$

$$g_2(\vec{\lambda}) = \inf_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1).$$

$x>0$ $1/x$ min is not defined

$x>0$ $1/x$ inf is 0



$\sup_t t \quad t \leq 1/x \ \forall x > 0$

$$g(\vec{\lambda}) = \underset{\vec{x}}{\cancel{\min}}\,\text{inf}\ \mathcal{L}(\vec{x}, \vec{\lambda})$$

$$\underset{\vec{x}}{\cancel{\min}}\,\text{inf}\ f_0(\vec{x}) + \lambda_1(-1-x) + \lambda_2(x-1)$$

$$\cancel{\min}\,\text{inf} \left\{ \underbrace{\underset{\vec{x}<0}{\cancel{\min}}\,\text{inf}\ \mathcal{L}(\vec{x}, \vec{\lambda})}_{g_2(\vec{\lambda})}, \underbrace{\underset{\vec{x}\geq 0}{\cancel{\min}}\,\text{inf}\ \mathcal{L}(\vec{x}, \vec{\lambda})}_{g_1(\vec{\lambda})} \right\}$$

iii. Next, show that

$$g_1(\vec{\lambda}) \le -3\lambda_1 + \lambda_2$$
$$g_2(\vec{\lambda}) \le \lambda_1 - 3\lambda_2.$$

$g_1(\lambda_1, \lambda_2) = \inf_{x \geq 0} h(x, \lambda_1, \lambda_2) \leq h(2, \lambda_1, \lambda_2)$

$h(x, \lambda_1, \lambda_2)$

Use this to show that $g(\vec{\lambda}) \le 0$ for all $\lambda_1, \lambda_2 \ge 0$.

$g_1(\vec{\lambda}) = \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1)$

$8 - 12 + 4 - 3\lambda_1 + \lambda_2$ $\quad x = 2$

$g_1(\vec{\lambda}) \leq -3\lambda_1 + \lambda_2$

$\min(-3\lambda_1 + \lambda_2, \lambda_1 - 3\lambda_2) \leq 0$

Plug $x = -2$

Similarly $g_2(\vec{\lambda}) \leq \lambda_1 - 3\lambda_2$

$g(\vec{\lambda}) = \min(g_1(\vec{\lambda}), g_2(\vec{\lambda})) \leq 0$

Some maths to show that there is no $\vec{\lambda} \geq 0$ such that $g_1(\lambda) > 0$ and $g_2(\lambda) > 0$

iv. Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

$\lambda_1 = 0 \quad \lambda_2 = 0 \quad g_1(\vec{0}) = 0$

$g_2(\vec{0}) = 0$ so $g(\vec{0}) = 0$

$g(\vec{0}) = 0 \quad \forall \vec{\lambda} \geq 0, \; g(\vec{\lambda}) \leq 0$

so $d^* = \max_{\vec{\lambda} \geq 0} g(\vec{\lambda}) = 0$

v. Does strong duality hold?

$p^* = 2 \quad d^* = 0 \quad \to$ No strong duality

(b) Now, consider a problem equivalent to the minimization in (1):

$$p^* = \inf_{x \in \mathbb{R}} f_0(x) \qquad (2)$$
$$\text{s.t. } \boxed{x^2 \le 1}$$

$-1 \leq x \leq 1 \Leftrightarrow x^2 \leq 1$

Observe that $\boxed{p^* = 2}$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part (a).

i. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda \ge 0} g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$g_1(\lambda) = \inf_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1)$$
$$g_2(\lambda) = \inf_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).$$

$$\mathcal{L}(x, \lambda) = f_0(x) + \lambda(x^2 - 1)$$

$$g(\lambda) = \min_x \mathcal{L}(x, \lambda) = \min\left(\min_{x \geq 0} \mathcal{L}(x,\lambda), \min_{x<0} \mathcal{L}(x,\lambda)\right)$$

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda))$$

continuous
show derivative is continuous

ii. Show that $\underbrace{g_1(\lambda) = g_2(\lambda)}$ = $\overbrace{\left\{ \begin{array}{ll} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3-\lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{array} \right.}^{g(\lambda)}$ $\quad h(x, \lambda)$

$x > 0 \quad = 0$

$$g_1(\lambda) = \min_{x \geq 0} \underbrace{x^3 - 3x^2 + 4 + \lambda(x^2 - 1)}_{C^1(\mathbb{R}, \mathbb{R})}$$

continuous and differentiable

opt. pt. such $\frac{\partial h}{\partial x}(\lambda, x) = 0$ for λ fixed

or $x = \pm\infty$ $\quad h(\pm\infty, \lambda) = +\infty$

You can do that

$3x^2 - 6x + 4 + 2\lambda x = 0 \to \dots \; x^*(\lambda)$

iii. Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

$g_1(\lambda) = h(x^*(\lambda), \lambda)$
$= g(\lambda)$

$$d^* = \max_{\lambda \geq 0} g(\lambda)$$

Case study : if $\lambda^* \geq 3$ $\quad \frac{dg}{d\lambda}(\lambda^*) = 0 \Rightarrow \lambda^* \dots$ compare with $g(+\infty)$

if $0 \leq \lambda^* \leq 3$ $\quad \frac{dg}{d\lambda}(\lambda^*) = 0 \Rightarrow \lambda^* \dots$ compare it with $g(0)$

iv. Does strong duality hold? $\quad \to g\left(\frac{3}{2}\right) = 2 = d^*$

$d^* = 2 \quad p^* = 2 \quad$ so strong duality holds

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$x \in [-1, 1]$ as $-1 \leq x \leq 1$ No strong duality

$x \in [-1, 1]$ as $x^2 \leq 1$ Strong duality

Duality if pb is not convex depends on the way to encode your feasible with constraints

Linear obj

$$\min_{\vec x} \; \boxed{\vec c^{\,T}\vec x}$$
$$\boxed{A\vec x \le \vec b}$$
$$C\vec x = \vec d$$

LP

Linear cnsts

$\{\vec x,\ \vec c^{\,T}\vec x = \alpha_1\}$ level set

$\vec x \subset \mathcal X \cap \{\vec x, \vec c^{\,T}\vec x = \alpha_3\}$

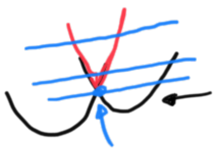
Solving

$$\arg\min_{\alpha} \quad 1$$
s.t. $\{\vec x, \vec c^{\,T}\vec x = \alpha\} \cap \mathcal X \neq \emptyset$

2. Linear programming

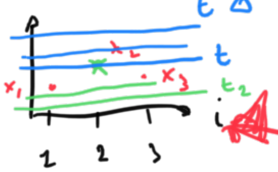
Express the following problems as LPs.

(a)



$$\min_{\vec x\in\mathbb{R}^k}\left[\max_{i=1,\ldots,k} x_i - \min_{j=1,\ldots,k} x_j\right]$$

s.t. $\quad A\vec x = \vec b$



$$\min_{\vec{\tilde x}} \; \vec{\tilde c}^{\,T}\vec{\tilde x}$$
$$\tilde A \vec{\tilde x} \le \vec{\tilde b}$$

$$\max_{i} x_i = \min t \qquad t \ge x_i \ \forall i$$

$$\max_{i=1,\ldots,k} x_i = \min t \qquad t \ge x_i \quad \forall i=1,\ldots,k$$

$$\min_{i=1,\ldots,k} x_i = \max t_2 \qquad t_2 \le x_i \quad \forall i=1,\ldots,k$$

$$\min_{\vec x} \left[\min_{t_1 \ge x_i} t_1 - \max_{t_2 \le x_i} t_2\right], \quad A\vec x = \vec b \qquad = \min_{\vec x} \min_{\ldots} t_1 + \min_{\ldots} -t_2$$

$$-\max t_2 = \min -t_2$$

(b)

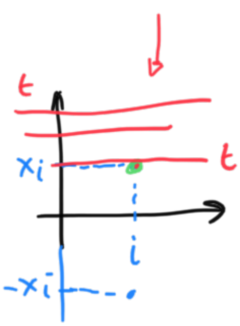
$$\min_{\vec x\in\mathbb{R}^k}\sum_{i=1}^{k}|x_i|$$

s.t. $\quad A\vec x = \vec b$

$$= \min_{\vec x, t_1, t_2} t_1 - t_2 \qquad = \vec{\tilde c}^{\,T}\vec{\tilde x}$$
$$A\vec x = \vec b \qquad = \tilde A \vec{\tilde x} = \vec{\tilde b}$$
$$t_1 \ge x_i \ \forall i \qquad = \vec d_1^{\,T}\vec{\tilde x} \le 0$$
$$t_2 \le x_i \ \forall i \qquad = \vec d_2^{\,T}\vec{\tilde x} \le 0$$

$$\vec{\tilde c}^{\,T} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{\tilde x} = \begin{pmatrix} \vec x \\ t_1 \\ t_2 \end{pmatrix}$$

$$\min_{\vec x \in \mathbb{R}^k} \sum_{i=1}^{k} |x_i|$$
$$A\vec x = \vec b$$



$$|x_i| = \max\{x_i, -x_i\} = \min t_i \qquad t_i \ge x_i \qquad t_i \ge -x_i$$

Because the minimization over t_i are independent from each other

$$\min_{\vec x \in \mathbb{R}^k} \sum_{i=1}^{k} \boxed{\min t_i \quad t_i \ge x_i \quad t_i \ge -x_i} \qquad A\vec x = \vec b$$

$$= \min_{\vec x \in \mathbb{R}^k} \min_{t_i, i=1,\ldots,k} \sum_{i=1}^{k} t_i$$
$$A\vec x = \vec b \qquad t_i \ge x_i \ \forall i \qquad t_i \ge -x_i \ \forall i$$

$$\min_x x + \boxed{\min_y y} = \min_{x,y} x + y$$

$$= \min_{\vec x \in \mathbb{R}^k, \vec t \in \mathbb{R}^k} \vec 1^{\,T}\vec t$$
$$A\vec x = \vec b$$
$$\vec t \ge \vec x$$
$$\vec t \ge -\vec x$$

LP