EECS 127 Optimization. Linear Algebra "Review"

Lecture 2. Jan 23, 2020.

Today: (1) Vectors, Norms.

(2) Gram-Schmidt, QR decomposition.

(3) Orthogonal Decomposition of a space

- Fundamental Thm. of Linear Algebra.

- Vector; Norms. $\vec{x}$ $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ $\vec{x} \in \mathbb{R}^n$

"Euclidean" Norm

Many others $f: \mathcal{X} \to \mathbb{R}$ $\mathcal{X}$: Vector space.

f is a norm if

(1) $\|\vec{x}\| \geq 0$ $\forall x \in \mathcal{X}$. And if $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$.

(2) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ $\forall x, y \in \mathcal{X}$. (Triangle Inequality).

(3) $\|\alpha \vec{x}\| = |\alpha| \cdot \|\vec{x}\|$ $\forall \alpha \in \mathbb{R}, \vec{x} \in \mathcal{X}$.

- l_p - norm.

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad 1 \le p < \infty$$

$p = 2 \rightarrow$ Euclidean norm.

$p = 1 \rightarrow \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$

$p = \infty \rightarrow \|\vec{x}\|_\infty = \max_{i=1,2,\ldots n} |x_i|$

Cauchy - Schwartz for 2-norm.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta$$

$$|\vec{x}^T \vec{y}| \le \|\vec{x}\|_2 \|\vec{y}\|_2$$

<u>Generalization:</u> Hölder's Inequality. $\vec{x}, \vec{y} \in \mathbb{R}^n$. $p, q \ge 1$.

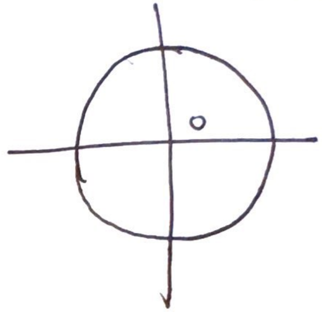
$$\frac{1}{p} + \frac{1}{q} = 1$$

$$|\vec{x}^T \vec{y}| \le \sum_{k=1}^{n} |x_k y_k| \le \|\vec{x}\|_p \|\vec{y}\|_q$$

- An optimization problem. $\vec{y} \in \mathbb{R}^n$

$$\max_{\|\vec{x}\|_p \le 1} \quad \vec{x}^T \vec{y}$$

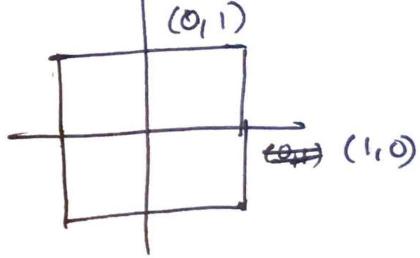
$\mathbb{R}^2$ $\quad \|\vec{x}\|_2 \le 1$



<u>$p=2$</u>

$$\max_{\|\vec{x}\|_2 \le 1} \quad \vec{x}^T \vec{y}.$$

$$\vec{x}^* = \frac{\vec{y}}{\|\vec{y}\|_2}$$



<u>$p=\infty$</u>

$$\vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$x_i = +1$ if $y_i \geq 0$ $\qquad \vec{x}^* = \operatorname{sgn}(\vec{y})$

$x_i = -1$ if $y < 0$

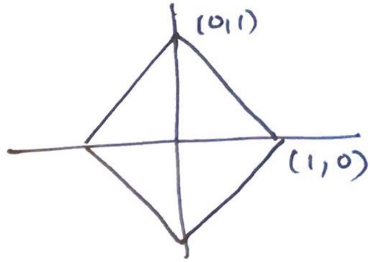
$$\max_{\|\vec{x}\|_\infty \le 1} \vec{x}^T \vec{y} = \sum |y_i| = \|\vec{y}\|_1$$

<u>$p=1$</u>

• $p = 1.$

$x_i = \cancel{?}\ \mathrm{sgn}(y_i)$ if y_i is the max. absolute value

$= 0$ otherwise.



$\|\vec{x}\|_1 \le 1.$

$$\max_{\|\vec{x}\|_1 \le 1} \vec{x}^T \vec{y} = \max_i |y_i|$$

$$= \|\vec{y}\|_\infty$$

(2) Gram-Schmidt / QR. $\quad \|\vec{x}\|_2$

Given a basis. $\{\vec{a_1}, \vec{a_2} \dots \vec{a_m}\}$ for some subspace

G.S. is a procedure to generate an orthonormal basis

that spans the same ~~sp~~ subspace.

(1) Choose $\vec{q_1} = \dfrac{\vec{a_1}}{\|\vec{a_1}\|_2}$ $\quad \|\vec{q_1}\|_2 = 1$

- Project $\vec{a_2}$ onto the span $\{\vec{q_1}\}$: $\vec{q_1} \langle \vec{a_2}, \vec{q_1} \rangle$

  Find the residual : $\vec{a_2} - \vec{q_1} \langle \vec{a_2}, \vec{q_1} \rangle = \vec{s_2}$

  Normalize $\vec{s_2}$: $\vec{q_2} = \dfrac{\vec{s_2}}{\|\vec{s_2}\|_2}$

- Project $\vec{a_3}$ onto the span $\{\vec{q_1}, \vec{q_2}\}$.

$$\vec{s_3} = \vec{a_3} - \langle \vec{a_3}, \vec{q_1} \rangle \cdot \vec{q_1} - \langle \vec{a_3}, \vec{q_2} \rangle \vec{q_2}.$$

$$\vec{q_3} = \frac{\vec{s_3}}{\|\vec{s_3}\|_2}$$

$A = QR$

Q : Orthogonal matrix

R : Upper-triangular matrix.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix}}_{Q} \begin{bmatrix} \|\vec{q}_1\| \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$\vec{a}_2 = \vec{s}_2 + \vec{q}_1 \langle \vec{a}_2, \vec{q}_1 \rangle = \|\vec{s}_2\|_2 \cdot \vec{q}_2 + \langle \vec{a}_2, \vec{q}_1 \rangle \cdot \vec{q}_1$$

$r_{11} = \|\vec{a}_1\|_2$

$r_{12} = \langle \vec{a}_2, \vec{q}_1 \rangle$

$r_{22} = \|\vec{s}_2\|_2$

$r_{13} = \langle \vec{a}_3, \vec{q}_1 \rangle$

$r_{23} = \langle \vec{a}_3, \vec{q}_2 \rangle$

$r_{33} = \|\vec{s}_3\|_2$

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

③ Orthogonal decompositions of a space.

$\mathcal{X}$: Vector space.

S: Subspace of $\mathcal{X}$.

Thm: $\vec{x} \in \mathcal{X}$. $\vec{x}$ can be written in a unique way as the sum of $\vec{s} \in S$, $\vec{r} \in S^{\perp}$

$S^{\perp}$ is the orthogonal complement of S.

$S^{\perp}$: $\{\vec{r} \mid \langle \vec{r}, \vec{s} \rangle = 0, \ \forall \vec{s} \in S\}$.

$$\mathcal{X} = S \oplus S^{\perp}$$

Proof: ① $S \wedge S^{\perp} = \{\vec{0}\}$.

$\vec{v} \in S, S^{\perp}$. $\langle \vec{v}, \vec{v} \rangle = \vec{0}$

② $W = S + S^{\perp}$. We will show. $W = \mathcal{X}$

- Choose an orthonormal basis for W, extend this basis to $\mathcal{X}$. Assume $\vec{z} \notin W$.

$\vec{z}$ in the basis $\perp W$,

$S \subseteq W$

$\vec{z} \perp W$, $\vec{z} \perp S$

$\Rightarrow \vec{z} \in S^{\perp} \subseteq W$

③ <u>Uniqueness</u>.

Consider.

$\vec{x_1}, \vec{x_2} \in S$, $\vec{y_1}, \vec{y_2} \in S^\perp$

$$\vec{x_1} + \vec{y_1} = \vec{x_2} + \vec{y_2}$$

But $\vec{x_1} \neq \vec{x_2}$

$\vec{y_1} \neq \vec{y_2}$

$$\Rightarrow \underbrace{\vec{x_1} - \vec{x_2}}_{\in S} = \underbrace{\vec{y_2} - \vec{y_1}}_{\in S^\perp}$$

$$\Rightarrow \vec{x_1} = \vec{x_2}$$

$$\vec{y_1} = \vec{y_2}$$